Week 8

Exercise 76 Suppose we've designed a new backpack for camping that is super light weight! On average the backpack weights 12 grams and the standard deviation is 1.1 grams. (Assume that each backpack are independent and that all distributions are normally distributed)

(1) What is the probability that a single backpack weighs between 11.8 and 12.2 grams?

(2) What is the probability that the average of 100 backpacks is between 11.8 and 12.2 grams?

Answer. Let X be the weight of a backpack and note that $E(X) = 12$ and $\text{SD}(X) = 1.1$.

(1) Since we are using the normal distribution, we estimate it using Φ:

$$\Phi\left(\frac{12.2-12}{1.1}\right) - \Phi\left(\frac{11.8-12}{1.1}\right) = \Phi\left(0.181818\right) - \Phi\left(-0.181818\right) = 0.57142 - 0.42858 = 0.14284$$

(2) Notice how we want to use the square root law we learned from week 6. If we add up all the backpacks we get S_{100}. Taking the average gives: $\bar{X}_{100} = \frac{S_{100}}{100}$. By the square root law we know

$$E(\bar{X}_{100}) = \frac{n \cdot E(X)}{n} = 12 \quad \text{and} \quad \text{SD}(\bar{X}_{100}) = \frac{\text{SD}(X)}{\sqrt{100}} = 0.11$$

So, we have

$$\Phi\left(\frac{12.2-12}{0.11}\right) - \Phi\left(\frac{11.8-12}{0.11}\right) = \Phi\left(1.81818\right) - \Phi\left(-1.81818\right) = 0.96485 - 0.03515 = 0.9297$$

□

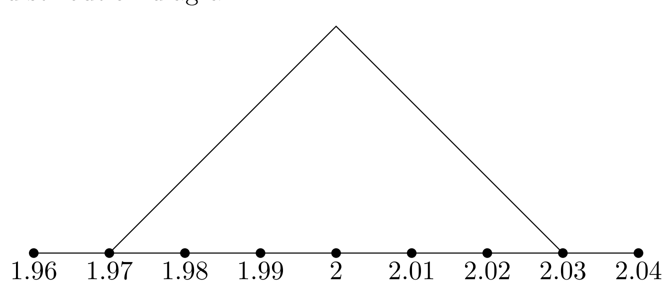
Exercise 77 Team Canada is going to the Olympics and they're getting ready to buy poles for pole vaulting! They go to a pole manufacturer and want to make sure they're being environmentally friendly by not throwing away to many "disfigured" poles. By regulation, the poles must be 2 meters long and are allowed to vary by at most 0.02 meters. (In other words, poles must be between 1.98 and 2.02 meters or else they are "disfigured" and are thrown away) Suppose that the expected value is equal to 2 meters and that the standard deviation is given by 0.01. What proportion of poles are disfigured and must be thrown away? (Assume a normal distribution)

Answer. We must throw away any poles that are less than 1.98 meters or more than 2.02 meters. So we need to calculate that probability. Since the expected value is 2 and our distribution is a normal distribution, we know that the distribution is symmetric about 2 and so we only need to calculate the value at 1.98.

$$2 \cdot \Phi\left(\frac{1.98-2}{0.01}\right) = 2 \cdot \Phi(-2) = 2 \cdot 0.02275 = 0.0455$$

So roughly 4.5% of poles from this manufacturer get thrown out. □

Exercise 78 Team Canada has decided that they want to actually check a second manufacturing company to see if they're more environmentally friendly or not. They go to a new company and the same principles apply. The poles must be 2 meters and anything outside of the 0.02 meter variance are all discarded. Unlike the previous company, the distribution is not a normal distribution. When asking about their distribution, you're given the following distribution diagram:



What proportion of poles are disfigured and must be thrown away.
Answer. Note the symmetry! So we really just need to calculate:

$$2\int_{1.97}^{1.98} f(x)\,\mathrm{dx}$$

where $f(x)$ is the function for the line on the left hand side of the triangle.

There are two ways to calculate this.

First the hard way: integration. In order to calculate what $f(x)$ is, we need to calculate the height of the triangle! We know that $\frac{1}{2}bh = 1$ where b is the base and h is the height (the 1 is coming from the fact that the total distribution must be equal to 1). We know that the base is equal to 0.06 implying that $h = \frac{1\cdot 2}{0.06} = 33.\bar{3}$. To calculate the slope we use:

$$\frac{\text{rise}}{\text{run}} = \frac{33.33333 - 0}{2 - 1.97} = \frac{10000}{9}$$

Using the point-slope formula at the point $(1.97, 0)$ we have:

$$y - y_0 = f(x) = \frac{10000}{9}(x - x_0) = \frac{10000}{9}x - \frac{10000}{9}\cdot 1.97 = \frac{10000}{9}x - \frac{19700}{9}$$

Plugging this into our integral we get:

$$\begin{aligned}
2\int_{1.97}^{1.98} \frac{10000}{9}x - \frac{19700}{9}\,\mathrm{dx} &= 2\cdot \frac{\frac{10000}{9}x^2}{2} - \frac{19700}{9}x\Bigg]_{1.97}^{1.98} \\
&= 2\cdot\left(\frac{10000\cdot(1.98^2 - 1.97^2)}{18} - \frac{19700\cdot(1.98 - 1.97)}{9}\right) \\
&= 2\cdot\left(\frac{10000\cdot 0.0395}{18} - \frac{19700\cdot 0.01}{9}\right) \\
&= 2\cdot\left(\frac{395}{18} - \frac{197}{9}\right) \\
&= 2\cdot(21.94444 - 21.8888) \\
&= 2\cdot 0.055555 \\
&= 0.1111111 \\
&= \frac{1}{9}
\end{aligned}$$

So roughly 11.1%.

Second the easy way: geometry. We just need to calculate the area under the curve from 1.97 to 1.98 and multiply by 2. We know that the height at 2 is $33.\bar{3}$; so since the line is straight, we know the height at 1.98 is $11.\bar{1} = \frac{100}{9}$. So we have

$$2\cdot\frac{1}{2}bh = 2\cdot\frac{1}{2}\cdot 0.01\cdot\frac{100}{9} = \frac{1}{9}$$

□

Exercise 79 Over time, paintings start to lose their colour as their pigments begin to fade. Suppose you just bought a painting and it has roughly 10^{20} pigments of colour. Each pigment fades in half roughly once every 10 years. (In other words, $P(T > 10) = \frac{1}{2}$) How many years until your painting only has 1000 pigments remaining? (Assume exponential distribution)
Answer. Since we're assuming exponential distribution, we know that $P(t < T) = e^{-\lambda t}$ and we are told $P(T > 10) = \frac{1}{2}$. Therefore, to calculate λ we have:

$$\frac{1}{2} = e^{-\lambda\cdot 10} \Rightarrow -\lambda\cdot 10 = \log\left(\frac{1}{2}\right) \Rightarrow \lambda = \frac{\log(2)}{10} = 0.069315$$

Since we only want 1000 out of 10^{20} to stay, that the same thing as saying 1 out of 10^{17} to stay. To find t, we can look at the probability $P(T > t) = \frac{1}{10^{17}}$ and solve for t:

$$P(T > t) = e^{-0.69315 \cdot t} = \frac{1}{10^{17}} \Rightarrow t = \frac{\log\left(10^{17}\right)}{\frac{\log(2)}{10}} \approx 564.727777$$

□

Exercise 80 You're given a light bulb that says the average life expectancy is 100 hours.

(1) What is the probability the light bulb lasts 200 hours?

(2) What is the standard deviation of the light bulb lasting?

(Assume exponential distribution)

Answer. We're told the average life expectancy is 100 hours. So let X be the life expectancy in hours and we know $E(X) = 100$. Therefore, the rate λ is given by $\frac{1}{100}$.

(1) The probability is given by

$$P(X > 200) = e^{-\frac{1}{100} \cdot 200} = e^{-2} \approx 0.135$$

(2) The standard deviation is given by

$$\text{SD}(X) = \frac{1}{\lambda} = 100$$

□

Exercise 81 Transistors that are produced by machine A have an average lifetime of 10 hours. Transistors that are produced by machine B have an average lifetime of 20 hours. Your company just received 12 transistors, 4 from machine A and 8 from machine B. Let X be the lifetime of a transistor picked at random from those received. What is the expected value and variance of X and what is the probability that the transistor lasts at least 20 hours? (Suppose that both machines have exponential distribution and are independent)

Answer. Let T_A be the lifetime of a transistor from machine A and let T_B be the lifetime of a transistor from machine B. We know $E(T_A) = 10$ and $E(T_B) = 20$ which implies T_A has parameter $\lambda_A = \frac{1}{10}$ and T_B has parameter $\lambda_B = \frac{1}{20}$. First, we're going to figure out the density function for X. Let X_A denote choosing a transistor from machine A and X_B denote choosing a transistor from machine B. Then

$$\begin{aligned} P(X > t) &= P(X > t \mid X_A)P(X_A) + P(X > t \mid X_B)P(X_B) \\ &= P(T_A > t)P(X_A) + P(T_B > t)P(X_B) \\ &= \frac{1}{3}e^{-\lambda_A t} + \frac{2}{3}e^{-\lambda_B t} \end{aligned}$$

The above gave us probabilities, but we want average, we can use a similar idea. We can just take the average of one thing and multiply it by the chance that thing will occur. Doing this over everything gives us what we want. Therefore, for $E(X)$ we have

$$E(X) = \frac{1}{3}E(T_A) + \frac{2}{3}E(T_B) = \frac{10}{3} + \frac{40}{3} = \frac{50}{3}$$

To calculate $\text{SD}(X)$ is a little more cumbersome. Recall that $\text{SD}(T_A) = 10$, but also that $\text{SD}(T_A) = \sqrt{E(T_A^2) - E(T_A)^2}$. This implies:

$$100 = E(T_A^2) - E(T_A)^2 = E(T_A^2) - 100 \Rightarrow 200 = E(T_A^2)$$

Similarly, $E(T_B^2) = 800$. Therefore, we have

$$\begin{aligned}
\mathrm{SD}(X) &= \sqrt{E(X^2) - E(X)^2} \\
&= \sqrt{\frac{1}{3}E(T_A^2) + \frac{2}{3}E(T_B^2) - \frac{50^2}{3^2}} \\
&= \sqrt{\frac{200}{3} + \frac{1600}{3} - \frac{2500}{9}} \\
&= \sqrt{\frac{600 + 4800 - 2500}{9}} \\
&= \sqrt{\frac{2900}{9}} \\
&= \frac{10\sqrt{29}}{3}
\end{aligned}$$

Finally, to calculate the probability that it lasts over 20 hours, we just need to plug-in 20 to our probability from above.

$$P(X > 20) = \frac{1}{3}e^{-\frac{20}{10}} + \frac{2}{3}e^{-\frac{20}{20}} = \frac{e^{-2}}{3} + \frac{2e^{-1}}{3} = 0.290365$$

□

Exercise 82 Suppose that we are building a new computer and we know that, on average, a new graphics card lasts 2 years (exponential distribution). We go to the store, buy 4 graphics cards and go home. The plan is to use one graphics card at a time, and when it goes bad, replace it with a new one. When we arrive home we find out that the bit coin miners have used up all the graphics cards and you can't buy any more for the next 10 years.

(1) What is the expected number of years your graphics cards will last?

(2) What is the standard deviation?

(3) What is the probability that your graphics cards will last 10 or more years?

Answer. Let C_i denote the random variable of the time the ith graphics card will last. We are told this is exponential. We let G_4 denote the time all four graphics cards last, *i.e.*, $G_4 = C_1 + C_2 + C_3 + C_4$. Since we have 4 graphics cards, we let the shape be given by $r = 4$. Since $E(C_i) = 2$, the scale parameter is given by $\lambda = \frac{1}{2}$.

Therefore we have

$$E(G_4) = \frac{r}{\lambda} = 4 \cdot 2 = 8$$

and

$$\mathrm{SD}(G_4) = \frac{\sqrt{r}}{\lambda} = 2 \cdot 2 = 4$$

The probability is given by

$$
\begin{aligned}
P(G_4 > 10) &= \int_{10}^{\infty} e^{-\lambda x} \frac{\lambda^r x^{r-1}}{(r-1)!}\,\mathrm{dx} \\
&= \int_{10}^{\infty} e^{\frac{-x}{2}} \frac{\frac{1}{2^4} \cdot x^3}{3!}\,\mathrm{dx} \\
&= \frac{1}{2^4 \cdot 6} \int_{10}^{\infty} e^{-x/2} x^3\,\mathrm{dx} \\
&= \frac{1}{16 \cdot 6} \left(-2e^{-x/2}x^3 - 12e^{-x/2}x^2 - 48e^{-x/2}x - 96e^{-x/2} \Big]_{10}^{\infty} \right) \quad \text{(int. by parts)} \\
&= \frac{1}{96} \left(2e^{-10/2}10^3 + 12e^{-10/2}10^2 + 48e^{-10/2}10 + 96e^{-10/2} \right) \\
&= \frac{1}{96} \left(3776e^{-5} \right) \\
&= \frac{118}{3} \cdot 0.006737946999 \\
&= 0.265025915297362
\end{aligned}
$$

□