Week 6

Exercise 57 Suppose that you go and buy some scratchers to test your luck. Of the $2,000$ scratchers, 250 of them are winners and the rest lose. You decide to buy 5 scratchers. (Scratchers are those lottery things that you have to scratch off to see if you win or not)

(1) What is the expected number of winning scratchers you will buy?

(2) What is an upper bound for the probability of having at least one winning scratcher?

(3) What is the exact probability of having at least one winning scratcher?

Answer. (1) Each scratcher has a $\frac{250}{2000} = \frac{1}{8}$ chance of winning. Let X_i be the indicator function on whether the ith scratcher we bought won or not. Then, let $X = \sum X_i$ be the number of winning scratchers we have. The expected number is then:

$$E(X) = \sum_{i=1}^{5} \frac{1}{8} = \frac{5}{8}$$

(2) Markov gives a nice inequality for this:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Since we want the upper bound for having at least one winning scratcher, we let $a = 1$ giving:

$$P(X \geq 1) \leq \frac{E(X)}{1} = E(X) = \frac{5}{8} = 0.625$$

(3) The actual probability is given by:

$$\begin{aligned} P(\text{at least one win}) &= 1 - P(\text{no wins}) \\ &= 1 - \frac{1750}{2000} \cdot \frac{1749}{1999} \cdot \frac{1748}{1998} \cdot \frac{1747}{1997} \cdot \frac{1746}{1996} \\ &= 1 - 0.512542 \\ &= 0.487458 \end{aligned}$$

The reason for this multiplication is that in order to calculate no wins, we need to actually buy five scratchers. Each time we grab a scratcher and look at whether we won or not, we're *given* new information and therefore the probabilities change. (Think of the birthday problem and the tree diagram we did for that.)

□

Exercise 58 Suppose you're waiting for the elevator in the Burj Khalifa. There are 163 floors! You walk into the elevator and 200 little kids rush into the elevator after you. (They have special elevators like in *Doctor who* that are bigger on the inside). Before you can get out, the doors close and you notice each kid start randomly (independently) choosing a random floor! You're now stuck, so you decide to just enjoy the view as the elevator goes up and stops a bunch of times. How many times do you expect the elevator to stop in total?

Answer. Let X be the number of floors at which the elevator stops. Obviously $E(X) \leq 163$, but we want to know the expected value. Each floor has a chance of being selected or not. So we can represent X as a sum of the 163 different floors and if they got selected or not. (aka indicator functions) So we have:

$$E(X) = \sum_{i=1}^{163} P(\text{at least one child chooses floor } i)$$

We just need to figure out the probability and do the sum. The probability that at least one child chooses a particular floor is given by:

$$\begin{aligned} P(\text{at least one child chooses floor } i) &= 1 - P(\text{no child chooses floor } i) \\ &= 1 - \left(\frac{162}{163}\right)^{200} \\ &\approx 1 - 0.2920667 \\ &= 0.7079333 \end{aligned}$$

In other words

$$E(X) \approx 163 \cdot 0.7079333 \approx 115.39313$$

□

Exercise 59 Suppose that I look at a standard deck of 52 cards which are randomly and fairly shuffled. Let X_1 be the number of cards which appear before the first ace. Let X_2 be the number of cards which appear before the second ace. Let X_3 and X_4 be the number of cards which appear before the third and fourth aces respectively. Let X_5 be the number of cards which appear after the last ace. Suppose that each X_i has the same distribution.

(1) What is the probability $P(X_i = k)$ for some value $0 \leq k \leq 48$.

(2) What is the expected value of each X_i? (Hint, don't use the probability)

(3) Are the X_i pairwise independent?

Answer. (1) Since each X_i has the same distribution, $P(X_i = k) = P(X_1 = k)$ for all i. Let's calculate $P(X_1 = k)$. We basically want the first k cards to *not* be an ace and then the $k+1$st card be an ace.

$$P(X_1 = k) = \frac{48}{52} \cdot \frac{47}{51} \cdot \dots \cdot \frac{48-k+1}{52-k+1} \cdot \frac{4}{52-k} = \frac{(48)_k \cdot 4}{(52)_{k+1}}$$

(2) Since expected value is linear, we get:

$$E(X_1 + \cdots + X_5) = E(X_1) + \ldots + E(X_5) = 5E(X_1)$$

We also know that

$$X_1 + X_2 + X_3 + X_4 + X_5 = 48$$

since they represent all cards that are not aces. So we also know that

$$E(X_1 + \cdots + X_5) = 48$$

Since we will *always* have all cards (that are not ace) when adding all of the X_i. Therefore:

$$E(X_1 + \cdots + X_5) = 5E(X_1) = 48 \quad \Rightarrow \quad E(X_i) = \frac{48}{5} = 9.6$$

(3) No! The location of the 2nd ace is dependent on the 1st ace (as an example). If the 1st ace is in the 10 position, the 2nd ace can't be in position 9. But if the 1st ace is in the 8th position, the 2nd ace *can* be in position 9.

□

Exercise 60 Difficult problem alert! Suppose that X is a random variable with potential values 0, 1 and 2. What is the distribution of X based on the expected values $E(X) = \mu$ and $E(X^2) = \xi$?

Answer. We know that $P(0) + P(1) + P(2) = 1$ by the Total one principle. We know $E(X) = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2)$ and $E(X^2) = 0 \cdot P(0) + 1 \cdot P(1) + 4 \cdot P(2)$. If you know linear algebra, this is an awesome place to use it! If not, we can just figure it out using our system of equations.

We have

$$\mu = E(X) = P(1) + 2P(2) \qquad \xi = P(1) + 4P(2)$$

So if we let $P(1) = \mu - 2P(2)$ then

$$\xi = E(X^2) = P(1) + 4P(2) = \mu - 2P(2) + 4P(2) = \mu + 2P(2)$$

In other words $P(2) = \frac{\xi-\mu}{2}$

This implies:

$$P(1) = \mu - 2P(2) = \mu - 2 \cdot \frac{\xi - \mu}{2} = \mu - \xi + \mu = 2\mu - \xi$$

and finally

$$P(0) = 1 - P(1) - P(2) = 1 - 2\mu + \xi - \frac{\xi - \mu}{2}$$

□

Exercise 61 Suppose that X represents the number of heads obtained if a (normal, fair) coin is tossed three times. Find the expected value and the variance of X^2.

Answer. For expected value of X^2 we have:

$$\begin{aligned} E(X^2) &= \sum x^2 P(X = x) \\ &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 4 \cdot P(X = 2) + 9 \cdot P(X = 3) \\ &= \frac{3}{8} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{1}{8} \\ &= \frac{3 + 12 + 9}{8} \\ &= \frac{24}{8} \\ &= 3 \end{aligned}$$

For the variance of X^2 we have:

$$\begin{aligned} \mathrm{Var}(X^2) &= E((X^2)^2) - E(X^2)^2 \\ &= \frac{3 + 48 + 81}{8} - 9 \\ &= \frac{132 - 72}{8} \\ &= \frac{60}{8} \\ &= \frac{15}{2} \end{aligned}$$

□

Exercise 62 Suppose that we have three different events: A, B, and C. Let $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{3}$. Let X be the total number of events A, B and C that occur.

(1) What is the expected value of X?

(2) If the events are disjoint, what is the variance?

(3) If the events are independent, what is the variance?

Answer. Since we're just counting the total number of times that the events occur, we can use indicator functions. Let X_A be the indicator function for A occurs; X_B and X_C are defined similarly. Then $X = X_A + X_B + X_C$.

(1) The expected value is then given by:

$$E(X) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \frac{12 + 15 + 20}{60} = \frac{47}{60}$$

(2) Let $D = A \cup B \cup C$ and notice that since our sets our disjoint, if an event happens, only *one* of the events will happen. In other words $X = X_A + X_B + X_C = X_D$ is an indicator function for D. Not only that, but since X is an indicator function, we can use the simplified formula for variance:

$$\mathrm{Var}(X) = P(D)\,(1 - P(D)) = E(X_D)\,(1 - E(X_D))$$

Therefore, the variance is given by

$$\mathrm{Var}(X) = E(X)\,(1 - E(X)) = \frac{47}{60}\left(1 - \frac{47}{60}\right) = \frac{47}{60} \cdot \frac{13}{60} = \frac{611}{3600}$$

(3) Since our events are independent, we know that we can sum up the variance. So:

$$\mathrm{Var}(X) = \mathrm{Var}(X_A) + \mathrm{Var}(X_B) + \mathrm{Var}(X_C)$$

As each of these is an indicator function, we can do like before:

$$\mathrm{Var}(X_A) = P(A)\,(1 - P(A)) = \frac{1}{5} \cdot \frac{4}{5} = \frac{4}{25}$$

repeating this process we get

$$\mathrm{Var}(X_B) = \frac{3}{16} \qquad \mathrm{Var}(X_C) = \frac{2}{9}$$

and

$$\mathrm{Var}(X) = \frac{4}{25} + \frac{3}{16} + \frac{2}{9} = \frac{2051}{3600}$$

□

Exercise 63 In Toronto, the average income is roughly \$65, 000. According to a report by Fong and Partners, you need an annual salary of \$135, 000 to be considered "middle class".

(1) Find an upper bound for the percentage of individuals who are considered at least middle class, *i.e.*, their income is over 135, 000.

(2) According to statcan, the standard of deviation of incomes for $25 - 29$ year olds, is \$30, 000. Can you find a better upper bound for the percentage of individuals (aged $25 - 29$) who are considered at least middle class?

(Hint: you need to use two different inequalities)
Answer. We let X be the income in thousands of dollars. So $E(X) = 65$ since the average/expected value of an income is \$65, 000, *i.e.*, 65 thousands of dollars.

(1) We can use Markov's inequality:

$$P(X \geq 135) \leq \frac{E(X)}{135} = \frac{65}{135} = 0.48148$$

In other words, at most 48% of individuals are middle class.

(2) We are told that the standard deviation is $\mathrm{SD}(X) = 30$. So we can use Chebychev's inequality! For this, we first need to calculate k. Since $X \geq 135$, then we need $|X - 65| \geq k \cdot 30$. In other words, $\frac{135-65}{30} \geq k$ which implies $k \leq \frac{7}{3}$. Therefore, let $k = \frac{7}{3}$, giving:

$$P(X \geq 135) \leq P(|X - 65| > 30 \cdot \frac{7}{3}) \leq \frac{1}{\left(\frac{7}{3}\right)^2} = \frac{3^2}{7^2} = 0.18367$$

In other words, at most 18% of individuals, aged $25 - 29$ are considered middle class.

□

Exercise 64 According to the book, in roulette, a "house special" is a bet on the numbers 0, 00, 1, 2 and 3 out of the 38 total possible numbers. So you have a $\frac{5}{38}$ chance of winning and for every dollar you bet, you get seven dollars back if you win (and nothing if you lose). If you make the "house special" bet 300 times where you bet $1 each time, what is the chance that you have at least as much money as you started with? (In other words, what's the probability you end with greater than or equal to $300.) Use normal approximation.
Answer. We have 300 bets and they're all independent of one another. Let X_i denote the ith bet. To calculate the total, we let S be the total after 300 bets, *i.e.*, $S = \sum X_i$. We want to know the probability that S is greater than or equal to 300, $P(S \geq 300)$. To use the central limit theorem, we need to find the expected value and the standard deviation. Since all the bets are independent, we just need to look at a particular bet.

So if we calculate the expected value after a particular bet we have:

$$E(X_i) = 7 \cdot \frac{5}{38} + 0 \cdot \frac{33}{38} = \frac{35}{38}$$

To calculate the variance we need:

$$E(X_i^2) = 7^2 \cdot \frac{5}{38} = \frac{245}{38}$$

Therefore, the variance is

$$\mathrm{Var}(X_i) = E(X_i^2) - E(X_i)^2 = \frac{245}{38} - \frac{35^2}{38^2} = \frac{8085}{1444}$$

giving us the standard deviation of

$$\mathrm{SD}(X_i) = \sqrt{\mathrm{Var}(X_i)} = \sqrt{\frac{8085}{1444}} = 2.366227$$

Putting this altogether we have

$$E(S) = 300 \cdot \frac{35}{38} = \frac{5250}{19} = 276.31579$$

and

$$\mathrm{SD}(S) = \sqrt{300 \cdot \frac{8085}{1444}} = 40.984255$$

Using the central limit theorem, we have:

$$P(S \geq 300) \approx 1 - \Phi\left(\frac{300 - 276.31579}{40.984255}\right) = 0.2817$$

□

Exercise 65 Suppose you were walking down the street and you found $100,000 just lying there with no one around. You pick it up and decide to keep it because, we're in a word problem so why not? You then decide to invest that money like a responsible adult. There are four different stock options, but you don't know which will give you a profit and which will take your money. You know one of them gives you a profit of 200 dollars for every 1000 you invest, another one gives 100 for every 1000 you invest, a third gives you nothing for every 1000 you invest and the last one makes you lose 100 for every 1000 you invest.

(1) Suppose you randomly choose to invest all $100,000$ into one stock. What is the probability that you will earn (at least) $8,000$ dollars?

(2) Suppose you decide to invest 1000 dollars, 100 times where each time you randomly chose a stock to invest in. What is the probability that you will earn (at least) $8,000$ dollars? (Use the central limit theorem)

Answer. (1) Since only two of the stocks will give you money, we only need to consider those two. If you get 100 for every 1000, then investing $100,000$ will give you $10,000$ and so both stocks which give you a profit will earn you over $8,000$. In other words, choosing those two out of the four possibilities will earn you money. Therefore, the probability is $\frac{1}{2}$.

(2) This case is more complicated (it is almost identical to the previous question). Each bet is independent and we want to know the sum of our profits after buying 100 stocks randomly. Let S be the sum of all of our profits and let S_i denote the ith purchase of a stock.

The expected value of a particular purchase is:

$$E(S_i) = 200 \cdot \frac{1}{4} + 100 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} - 100 \cdot \frac{1}{4} = 50$$

and we can see

$$E(S_i^2) = 200^2 \cdot \frac{1}{4} + 100^2 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + (-100)^2 \cdot \frac{1}{4} = 15,000$$

Therefore, our standard deviation is

$$\text{SD}(S_i) = \sqrt{\text{Var}(S_i)} = \sqrt{E(S_i^2) - E(S_i)^2} = \sqrt{15,000 - 50^2} = \sqrt{12500} = 111.8034$$

Looking at the sum of all of our profits, we get:

$$E(S) = 100E(S_1) = 5000 \qquad \text{SD}(S) = \sqrt{100}\,\text{SD}(S_1) = 10 \cdot 111.8034 = 1118.034$$

Using the central limit theorem we get

$$P(S \geq 8000) = 1 - \Phi(\frac{8000 - 5000}{1118.034}) = 1 - \Phi(2.6833) = 1 - 0.9963 = 0.0037$$

□

Exercise 66 Let D_i be a random digit between 0 and 9. Let X_i be the last digit of D_i^2 (so if $D_i = 8$ then $D_i^2 = 64$ and $X_i = 4$). Let $\bar{X}_n = \frac{X_1 + \ldots + X_n}{n}$ be the average of a large number n where each X_i is determined from randomly chosen D_i.

(1) Predict the value of $\bar{X}_n$ for large n.

(2) If you just had to predict the last digit of $\bar{X}_{100}$, what digit should you chose to maximize your chance of being right, and what is the probability?

Answer. First we need to figure out the distribution of our probability. We have ten digits:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

Squaring each one gives:

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81$$

Since we'll be adding numbers together, we keep our sample space to be all of the digits. Therefore our distribution of the X_i is given by:

$$P(X_i = 0) = \frac{1}{10},\ P(X_i = 1) = \frac{2}{10},\ P(X_i = 4) = \frac{2}{10}$$
$$P(X_i = 5) = \frac{1}{10},\ P(X_i = 6) = \frac{2}{10},\ P(X_i = 9) = \frac{2}{10}$$

and the rest are 0. So our expected value, variance and standard deviation are:

$$E(X_i) = 0 \cdot \frac{1}{10} + 1 \cdot \frac{2}{10} + 4 \cdot \frac{2}{10} + 5 \cdot \frac{1}{10} + 6\frac{2}{10} + 9 \cdot \frac{2}{10} = 4.5$$
$$E(X_i^2) = 0^2 \cdot \frac{1}{10} + 1^2 \cdot \frac{2}{10} + 4^2 \cdot \frac{2}{10} + 5^2 \cdot \frac{1}{10} + 6^2\frac{2}{10} + 9^2 \cdot \frac{2}{10} = \frac{293}{10}$$
$$\text{Var}(X_i) = E(X_i^2) - E(X_i)^2 = 29.3 - 4.5^2 = 9.05$$
$$\text{SD}(X_i) = \sqrt{\text{Var}(X_i)} = \sqrt{9.05} = 3.008322$$

(1) By the law of averages, as n goes to infinity the difference between $\bar{X}_n$ and $E(X)$ goes to 0 where X is one of the X_i. Therefore, we can estimate this as:

$$\bar{X}_n \approx E(X) = E(X_i) = 4.5$$

(2) This question should remind you of the mode! So if you think about it that way, taking the floor, we'd expect 4 to be the value of the last digit. To calculate the probability, what we want to know is the probability $P(4 \leq \bar{X}_{100} < 5)$. Let $S_n = X_1 + \ldots + X_n$, *i.e.*, $\bar{X}_n = \frac{S_n}{n}$. Recall that for the central limit theorem we have

$$P\left(a \leq \frac{S_n - nE(X)}{\sqrt{n}\,\text{SD}(X)} \leq b\right) \approx \Phi(b) - \Phi(a)$$

Therefore, we want to convert our probability to look like the above:

$$\begin{aligned}
P(4 \leq \bar{X}_{100} < 5) &= P\left(4 \leq \frac{S_{100}}{100} < 5\right) \\
&= P\left(4 \cdot 100 \leq S_{100} < 5 \cdot 100\right) \\
&= P\left(400 - 100 \cdot 4.5 \leq S_{100} - 100 \cdot 4.5 < 500 - 100 \cdot 4.5\right) \\
&= P\left(\frac{400 - 450}{\sqrt{100} \cdot 3.008322} \leq \frac{S_{100} - 100 \cdot 4.5}{\sqrt{100}3.008322} < \frac{500 - 450}{\sqrt{100} \cdot 3.008322}\right) \\
&= P\left(\frac{-50}{10 \cdot 3.008322} \leq \frac{S_{100} - 100 \cdot E(X)}{\sqrt{100}\,\text{SD}(X)} < \frac{50}{10 \cdot 3.008322}\right) \\
&= P\left(-1.662234 \leq \frac{S_{100} - 100 \cdot E(X)}{\sqrt{100} \cdot \text{SD}(X)} < 1.662234\right) \\
&\approx \Phi(1.662234) - \Phi(1.662234) \\
&= 2\Phi(1.662234) - 1 \\
&= 2 \cdot 0.9515 - 1 \\
&= 0.903
\end{aligned}$$

□