Week 3

Exercise 31 Suppose that in 6-person families, every person has an equal chance of being male, female or non-binary independently of one another. Which would be more common:

- families with exactly 3 females
- families without exactly 3 females?

Answer. Since each individual has $\frac{1}{3}$ of being female, we know that $p = \frac{1}{3}$. Since there are six individuals, we know $n = 6$. To find the number of families with exactly 3 females we set $k = 3$ and solve using the binomial distributions.

$$P(3) = \binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 = \frac{6!}{3!3!}\cdot\frac{2^3}{3^6} = \frac{6\cdot 5\cdot 4}{3\cdot 2}\cdot\frac{2^3}{3^6} = \frac{5\cdot 2^5}{3^6} = 0.2195$$

The other probability is given by $1 - P(3) = 0.7805$ therefore it's more likely to have a family without 3 females. □

Exercise 32 Say I flip a coin 8 times and 3 of them turn out to be heads. What are the chances that 2 of the heads appeared in the first 6 flips?

Answer. We are asking what the probability of "2 heads in 6 flips given 3 heads in 8 flips". So this is equal to the probability of "2 heads in 6 flips and 3 heads in 8 flips" divided by the probability of "3 heads in 8 flips".

So let's first calculate "3 heads in 8 flips". For this $n = 8$, $p = \frac{1}{2}$ and $k = 3$. So we have

$$\text{Probability } = \binom{8}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 = \frac{8!}{3!5!}\cdot\frac{1}{2^8} = \frac{8\cdot 7\cdot 6}{3\cdot 2}\cdot\frac{1}{2^8} = \frac{7}{2^5} = 0.21875$$

To figure out "2 heads in 6 flips and 3 heads in 8 flips", we need to notice that this is equal to "2 heads in 6 flips and 1 flip in the next 2 flips. These are now independent events! So we calculate each individually:

$$n = 6,\ k = 2 \ \Rightarrow\ \frac{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{6\cdot 5}{2^7} = 0.234375$$

$$n = 2,\ k = 1 \ \Rightarrow\ \frac{2}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1 = \frac{2}{2^2} = 0.5$$

Therefore we're left with:

$$\frac{0.234375\cdot 0.5}{0.21875} = 0.535714$$

□

Exercise 33 A human is playing darts and has super good aim. They're able to hit the red bullseye with probability 0.7. Assume each throw is independent and that he makes 8 throws in total.

(1) Given that they hit the bullseye at least twice, what is the chance that they hit the bullseye exactly four times?

(2) Given that the first two throws hit the bullseye, what is the chance that they hit the bullseye exactly four times in the 8 throws?

Answer. (1) The probability of the event is given by:

$$\begin{aligned}
P(\text{exactly 4 bullseyes} \mid \text{at least 2 bullseyes}) &= \frac{P(\text{exactly 4 bullseyes and at least 2 bullseyes})}{P(\text{at least 2 bullseyes})} \\
&= \frac{P(\text{exactly 4 hits})}{1 - (P(\text{exactly 0 bullseyes}) + P(\text{exactly 1 bullseyes}))} \\
&= \frac{\binom{8}{4}(0.7)^4(0.3)^4}{1 - \binom{8}{0}(0.7)^0(0.3)^8 - \binom{8}{1}(0.7)^1(0.3)^7} \\
&= \frac{\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}\cdot(0.7)^4(0.3)^4}{1-(0.3)^8 - 8(0.7)(0.3)^7} \\
&= \frac{0.1361367}{0.99870967} \\
&= 0.1363126
\end{aligned}$$

The second equality comes from the fact that "exactly 4 bullseyes" is a subset of "at least 2 bullseyes". So converting "and" to $\cap$ we get, "exactly 4 bullseyes" $\cap$ "at least 2 bullseyes" = "exactly 4 bullseyes".

(2) This is the same thing as asking whether we get exactly two bullseyes in the final 6 throws. So we get:

$$\begin{aligned}
P(A) &= \binom{6}{2}(0.7)^2(0.3)^4 \\
&= \frac{6\cdot5}{2}(0.7)^2(0.3)^4 \\
&= 0.059535
\end{aligned}$$

□

Exercise 34 A battle for the royal dice thrower is about to begin. Two humans are sat in front of one another with a fair eight sided die in front of them. Each round, the two humans roll the die. If one person scores higher than another, they win the round; if the score is even, it's a tie. There are five rounds in this tournament. What are the chances that the first player wins at least four out of five rounds?

Answer. We must decide what the probability of winning a round is. The two dice rolls can be written as (i, j). So the first player wins the round if $i > j$. If $j = 1$ there are 7 numbers greater than j; $j = 2$ there are 6, etc. So there are $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ ways to win. There are $8 \cdot 8$ different possibilities in total (sequences of length 2 of 8 elements) and so the probability for them to win is given by: $\frac{28}{64} = \frac{7}{16}$.

Then, using the binomial distribution, winning at least four times is given by:

$$\begin{aligned}
P(4) + P(5) &= \binom{5}{4}\left(\frac{7}{16}\right)^4\left(\frac{9}{16}\right)^1 + \binom{5}{5}\left(\frac{7}{16}\right)^5 \\
&= \frac{5\cdot7^4\cdot9}{16^5} + \frac{7^5}{16^5} \\
&= 0.119068
\end{aligned}$$

□

Exercise 35 Roughly 65% of Toronto have received their adulting cards in the mail. Say that we randomly select a sample of 20 individuals from Toronto.

(1) What is the most likely number of people with adulting cards in the sample?

(2) What is the chance of getting this many people with adulting cards?

Answer. We are told that $p = 0.65$ and $n = 20$.

(1) The first question is asking for the mode. The mode is given by:

$$m = \lfloor np + p \rfloor = \lfloor 20 \cdot 0.65 + 0.65 \rfloor = \lfloor 13.65 \rfloor = 13$$

(2) The chance of getting exactly 13 people out of 20 is given by the binomial distribution:

$$\begin{aligned} P(13) &= \binom{20}{13} (0.65)^{13} (0.35)^{7} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} (0.65)^{13} (0.35)^{7} \\ &= 19 \cdot 17 \cdot 16 \cdot 15 \cdot (0.65)^{13} (0.35)^{7} \\ &= 0.1844 \end{aligned}$$

□

Exercise 36 In a hospital, there are 300 patients waiting to get an untested treatment. The patients have (independently) a $\frac{1}{3}$ chance of surviving the treatment. Using the normal approximation, calculate the probability that more than 120 people will live after receiving treatment.

Answer. The expected value is given by $\mu = np = \frac{300}{3} = 100$. The standard deviation is given by $\sigma = \sqrt{np(1-p)} = \sqrt{100 \cdot \frac{2}{3}} = 8.16$. If S is the number of survivors, then

$$\begin{aligned} P(S > 120) &\approx 1 - \Phi\left(\frac{121 - \frac{1}{2} - \mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{120.5 - 100}{8.16}\right) \\ &= 1 - \Phi(2.51) \\ &= 0.006 \end{aligned}$$

□

Exercise 37 You decided to gamble away your life savings of \$25 and so you went to a casino and decided to play roulette. You don't want to lose your savings to fast, so instead you decide to bet a dollar on red 25 times. If the ball lands on red, you get \$2. The probability of getting a red is $\frac{18}{38}$. Find the normal approximation that after 25 bets, you will have \$25 or more.

Answer. We know $\mu = np = \frac{25 \cdot 18}{38} = 11.84$ and $\sigma = \sqrt{np(1-p)} = \sqrt{11.84(\frac{20}{38})} = 2.50$. In order to have 25 or more you must win 13 or more times. So

$$\begin{aligned} P(W \geq 13) &\approx 1 - \Phi\left(\frac{13 - \frac{1}{2} - \mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{0.66}{2.5}\right) \\ &= 1 - \Phi(0.264) \\ &= 0.3974 \end{aligned}$$

□

Exercise 38 Suppose that you're looking at reviews of a product that you just released. You know that roughly 45% of the people who have bought the product actually enjoyed it. You took a poll and 200 people responded back (independently and randomly and equally likely).

(1) Estimate the chance that exactly 90 of the people who responded back actually enjoyed the product.

(2) Estimate the chance that over 50% of the people who responded back actually enjoyed the product.

(Use normal approximation)

Answer. We have $n = 200$ and $p = 0.45$, so $\mu = 200 \cdot 0.45 = 90$ and $\sigma = \sqrt{90 \cdot 0.55} = 7.035$. Then

(1) The chance is given by:

$$\Phi\left(\frac{90+0.5-90}{7.035}\right) - \Phi\left(\frac{90-0.5-90}{7.035}\right) = 0.5279 - 0.4721 = 0.0558$$

(2) The chance is given by:

$$1 - \Phi\left(\frac{101-0.5-90}{7.035}\right) = 1 - \Phi(1.49) = 1 - 0.9319 = 0.0681$$

□

Exercise 39 Using the normal approximation, find the chance of getting 100 sixes in 600 (fair) dice rolls (of a six sided die).

Answer. We know that $n = 600$, $p = \frac{1}{6}$, $\mu = 100$, and $\sigma = \sqrt{100 \cdot \frac{5}{6}} = 9.1287$. For normal approximation we do:

$$\begin{aligned}
\Phi\left(\frac{100+\frac{1}{2}-\mu}{\sigma}\right) - \Phi\left(\frac{100-\frac{1}{2}-\mu}{\sigma}\right) &= \Phi\left(\frac{1}{2\sigma}\right) - \Phi\left(\frac{-1}{2\sigma}\right) \\
&= \Phi(0.05477) - (1 - \Phi(0.05477)) \\
&= 2\Phi(0.05477) - 1 \\
&= 2 \cdot 0.5199 - 1 \\
&= 1.0398 - 1 \\
&= 0.0398
\end{aligned}$$

□

Exercise 40 Suppose we ran a test with 30 individuals in which we had them flip a (fair) coin 200 times. What (approximately) are the chances that none of the individuals flip exactly 100 heads?

Answer. First we figure out what are the chances of any one of them flipping exactly 100 heads. We know $n = 200$, $p = 0.5$, $\mu = 100$ and $\sigma = 7.07$. Then, we approximate:

$$P(100) = 2\Phi\left(\frac{1}{2 \cdot \sigma}\right) - 1 = 2 \cdot 0.5282 - 1 = 0.0564$$

(This approximation uses the exact same technique as in Exercise 39)

We want to know the chances that *none* of the individuals flip exactly 100 heads. This is the same probability as *all* individuals not flipping exactly 100 heads.

$$(P(\text{not } 100))^{30} = (1 - P(100))^{30} = (1 - 0.0564)^{30} = 0.175$$

□