Week 2

Exercise 18 Suppose we are looking at a group of humans that are either right or left handed and with either green or blue hair. If 88% of blue-haired individuals are right handed and 78% of green-haired people are right handed then which of the following are true? Which are false? Which can't be decided with the given information?

(1) The overall proportion of right handers in this group is exactly 83%.

(2) The overall proportion of right handers in this group is between 88 and 78 percent.

(3) If the ratio of blue-haired humans to green-haired humans is 1 to 1 then the overall proportion of right handers in this group is exactly 83%.

Answer. (1) Can't tell. We don't know enough information to be able to answer this question.

(2) True. If there are no blue-haired individuals, then we have 78% and if there are no green-haired individuals, then we have 88%. All other options are in-between these two.

(3) True. If we have 1 to 1 ratio then our odds are 1 : 1 which means there are 50% blue-haired individuals and 50% green-haired individuals. So the law of total probability gives:

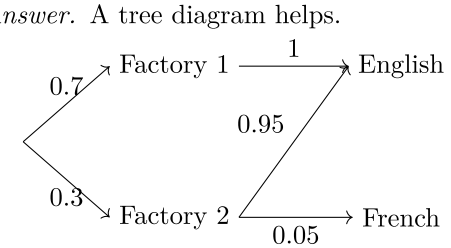
$$P(R) = P(R \mid B)P(B) + P(R \mid G)P(G) = 0.78 \cdot 0.5 + 0.88 \cdot 0.5 = 0.83$$

where R is the event "right-handed", B is "blue-haired", and G is "green-haired".

□

Exercise 19 Suppose a book printing company has two factories. The first factory produces 70% of the company's books (all in English). The second factory produces the rest of the books. Most of the books in the second factory are printed in English, but 5% are produced in French. Among all the book printed (combined) from both factories what proportion are printed in French?

Answer. A tree diagram helps.



Let A be the event "Printed in factory one", implying A^c is the event "Printed in factory two". We let E be the event "Printed in English" and E^c be the event "Printed in French.

The question is answered by:

$$P(A \cap E^C) + P(A^c \cap E^C) = 0 + P(A^c)P(E^c|A^c) = 0.3 \cdot 0.05 = 0.015$$

Therefore 1.5% of the books are printed in French. □

Exercise 20 Suppose that two independent events have probabilities 0.2 and 0.6 of occurring.

(1) What is the probability of neither of the events occurring?

(2) What is the probability at least one of the events occurs?

(3) What is the probability exactly one of the events occurs?

Answer. Suppose that $P(A) = 0.2$ and $P(B) = 0.6$ to make life easier.

(1) Neither event occurring is given by

$$P(A^c \cap B^c) = P(A^c)P(B^c) = 0.8 \cdot 0.4 = 0.32$$

(2) At least one event occurring means that two events might occur. So we want the complement of the previous question. So we get:

$$1 - P(A^c \cap B^c) = 1 - 0.32 = 0.68$$

(3) To know when *exactly* one event occurs, we can look at the following:

$$P((A \cap B^c) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c + B) = P(A)P(B^c) + P(A^c)P(B) = 0.2 \cdot 0.4 + 0.8 \cdot 0.6 = 0.56$$

Another way to calculate this is to look at when exactly both events occur:

$$P(A \cap B) = P(A)P(B) = 0.2 \cdot 0.6 = 0.12$$

and remove it from the previous question:

$$0.68 - 0.12 = 0.56$$

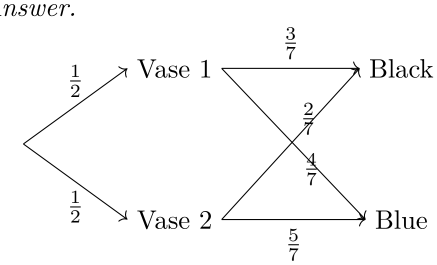
□

Exercise 21 Say we have two vases where the first vase contains 3 black balls and 4 blue balls and the second vase contains 2 black balls and 5 blue balls. We first randomly choose a vase and then randomly choose a ball.

(1) Draw a tree diagram.

(2) Calculate the probability that the ball drawn is blue.

Answer.



The probability that it's blue is given by:

$$\frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{5}{7} = \frac{9}{14}$$

□

Exercise 22 Suppose that we are dealt two cards from a standard deck of 52 cards. What is the probability that the second card is a diamond knowing that the first one is red?

Answer. We are in essence asking $P(\text{second draw diamond} \mid \text{first draw red})$. Let D = second draw diamond and let R = first draw red. Therefore

$$P(D \mid R) = \frac{P(D \cap R)}{P(R)}$$

Now "red" can mean either diamond or heart, so we need to split the two. Let R_d = first draw diamond and let R_h = first draw heart. Then

$$\frac{P(D \cap R)}{P(R)} = \frac{P(D \cap R_d)}{P(R)} + \frac{P(D \cap R_h)}{P(R)} = \frac{(13/52)(12/51)}{(26/52)} + \frac{(13/52)(13/51)}{(26/52)} = \frac{25}{102}$$

□

Exercise 23 Suppose I have a deck of cards where

- 15% are red on both sides
- 65% are red on one side and black on the other

- 20% are black on both sides

I've randomly shuffled the deck and the top card is red on top. What is the probability that the other side of the top card is black?
Answer. Since each card has two sides, we really need to consider these as $2n$ different cards since "red on both sides" doesn't tell us *which* side is up. (Think of this as first we take our deck right side up and then up-side down, giving us $2n$ options)

$$P(\text{black on bottom} \mid \text{red on top}) = \frac{P(\text{black on bottom} \cap \text{red on top})}{P(\text{red on top})} = \frac{0.65 \cdot 0.5}{0.65 \cdot 0.5 + 0.2} = 0.619$$

(These half are coming from the fact that 65% of the cards are red on one side and black on the other. So when we look at the two sides, half will have red on top and black on the bottom and half will have red on the bottom and black on top.) □

Exercise 24 Suppose that two independent internet companies supply internet to your neighbourhood. To save money they only supply internet sometimes. The first company supplies internet with probability 0.2 and the second with probability 0.7. If both companies supply internet then the whole neighbourhood has internet with probability 1. If only one company supplies internet then the whole neighbourhood has internet with probability 0.4. If none supply internet, then there is no internet.

(1) What is the probability of both companies supplying internet?

(2) What is the probability of exactly one company supplying internet?

(3) What is the probability of no company supplying internet?

(4) What is the probability that your neighbourhood has internet?

Answer. Let A be the first company and B be the second company.

(1) We are asking when both work:

$$P(A \cap B) = P(A)P(B) = 0.2 \cdot 0.7 = 0.14$$

(2) Exactly one working is given by:

$$P(A \cap B^c) + P(A^c \cap B) = P(A)P(B^c) + P(A^c)P(B) = 0.2 \cdot 0.3 + 0.8 \cdot 0.7 = 0.62$$

(3) None of them working is given by:

$$P(A^c \cap B^c) = P(A^c)P(B^c) = 0.8 \cdot 0.3 = 0.24$$

(Alternatively: $1 - (0.62 + 0.14) = 1 - 0.76 = 0.24$)

(4) We now just need to multiply probabilities:

$$P(A \cap B) \cdot 1 + (P(A \cap B^c) + P(A^c \cap B)) \cdot 0.4 + P(A^c \cap B^c) \cdot 0 = 0.14 + 0.62 \cdot 0.4 + 0 = 0.388$$

□

Exercise 25 How many people must be present in a room for there to be at least a 50% chance that two or more of them were born in the same month?
Answer. This is similar to the birthday problem. We organize everyone into some order and we go one at a time. We end up with the formula:

$$1 - \frac{11}{12} \cdot \frac{10}{12} \cdot \ldots \cdot \frac{12 - (n-1)}{12}$$

when there are n people in the room.

When $n = 2$ we have

$$1 - \frac{11}{12} = \frac{1}{12} < \frac{1}{2}$$

When $n = 3$ we have

$$1 - \frac{11}{12}\frac{10}{12} = \frac{17}{72} < \frac{1}{2}$$

When $n = 4$ we have

$$1 - \frac{11}{12}\frac{10}{12}\frac{9}{12} = \frac{123}{288} < \frac{1}{2}$$

When $n = 5$ we have

$$1 - \frac{11}{12}\frac{10}{12}\frac{9}{12}\frac{8}{12} = \frac{178}{288} > \frac{1}{2}$$

So we need 5 people in the room to have a 50% chance. □

Exercise 26 Suppose you roll a fair six sided dice until you roll a number that's already been rolled.

(1) Let p_i to be the probability that you rolled *exactly* i times before succeeding. Calculate p_i for the numbers 1 until 10.

(2) Without any calculations find $p_1 + p_2 + \cdots + p_{10}$ and explain why you found that solution.

Answer. (1) Think of this as we look at the first i rolls that must be different and then find a duplicate.

(a) $p_1 = 0$
(b) $p_2 = \frac{1}{6}$
(c) $p_3 = \frac{5}{6}\frac{2}{6} = \frac{10}{36}$
(d) $p_4 = \frac{5}{6}\frac{4}{6}\frac{3}{6} = \frac{60}{216}$
(e) $p_5 = \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{4}{6} = \frac{240}{1296}$
(f) $p_6 = \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{5}{6} = \frac{600}{7776}$
(g) $p_7 = \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{1}{6}\frac{6}{6} = \frac{120}{7776}$
(h) $p_8 = p_9 = p_{10} = 0$

(2) $p_1 + p_2 + \ldots + p_{10} = 1$ since by the 7th spin we're guaranteed to have hit two dice which are the same number.

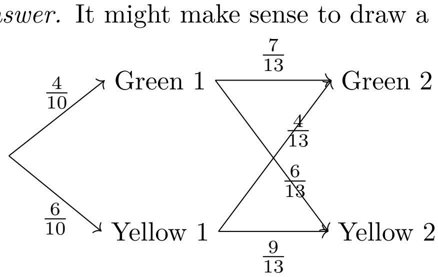
□

Exercise 27 Suppose that we have a vase with 4 green balls and 6 yellow balls. We chose a ball at random, write down the colour and then put it back into the vase. We then take three balls of the written down colour and add it into the vase. (So if I had drawn a green ball, I would put the green ball back in and add three more green balls making the vase have 7 green balls and 6 yellow balls). We then pull another ball randomly from the vase.

(1) What is the probability that the second ball is yellow?

(2) What is the probability that the first ball was green if the second ball was yellow?

Answer. It might make sense to draw a tree diagram to help.



Let "yellow$_2$" represent pulling a yellow for the second ball and let "green$_1$" represent pulling a green as a first ball.

(1) Using a tree diagram we know that

$$P(\text{yellow}_2) = \frac{4}{10} \cdot \frac{6}{13} + \frac{6}{10} \cdot \frac{9}{13} = \frac{24+54}{130} = \frac{78}{130}$$

(2) We have

$$P(\text{green}_1|\text{yellow}_2) = \frac{P(\text{green}_1 \cap \text{yellow}_2)}{P(\text{yellow}_2)} = \frac{\frac{4}{10} \cdot \frac{6}{13}}{\frac{78}{130}} = \frac{24}{78} = \frac{4}{13}$$

□

Exercise 28 You're in charge of testing a new fibre optics cable that your company has manufactured. The cable transmits data from one end to the other. The transmitting side sends bursts of light which represent either a zero or a one to the receiving side. Since cables aren't perfect, there is some loss of data from one side to the other. You know that the probability that the receiving end receives a zero when the transmitter sends a zero is 99%. You know that the probability that the receiving end receives a one when the transmitter sends a one is 98%. You also know that the transmitting end sends a one exactly 50% of the time.

(1) What is the probability that the transmitter sent a 0 if the receiver received a 1?

Answer. We let T_0 represent "transmit a 0", T_1 represent "transmit a 1", R_0 represent "receive a 0" and R_1 represent "receive a 1". We are given that $P(R_0 \mid T_0) = 0.99$, $P(R_1 \mid T_1) = 0.98$ and $P(T_1) = 0.5$. This implies $P(R_0^c \mid T_0) = P(R_1 \mid T_0) = 0.01$, $P(R_0 \mid T_1) = 0.02$ and $P(T_0) = 0.5$.

(1) We want to find $P(T_0 \mid R_1)$.

$$P(T_0 \mid R_1) = \frac{P(T_0 \cap R_1)}{P(R_1)} = \frac{P(T_0)P(R_1 \mid T_0)}{P(R_1 \mid T_0)P(T_0) + P(R_1 \mid T_1)P(T_1)} = \frac{0.01 \cdot 0.5}{(0.01 \cdot 0.5) + (0.98 \cdot 0.5)} = \frac{1}{99}$$

□

Exercise 29 You're not feeling to well one day and decide to go to the hospital to see if you have the flu. You know that, generally, 1% of the population has the flu at any given time. You decide to take a flu test to see if you really have the flu or not. When you go into the hospital they say that a false positive (a person who isn't sick getting a positive result) happens about 5% of the time and a false negative (a person who is sick gets a negative result) happens about 20% of the time.

(1) What are the chances that the test will come back positive? (No matter if you have the flu or not)

(2) What are the chances that you have the flu and the test comes back negative?

(3) What are the chances that the test shows positive, but you don't have the disease?

(4) Is this a frequency or a subjective interpretation?

Answer. We let p means "positive test result", n to mean "negative test result", and f to mean "has the flu". We are given that $P(f) = 0.01$, $P(p \mid f^c) = 0.05$ and $P(n \mid f) = 0.2$. This implies $P(f^c) = 0.99$, $P(n \mid f^c) = 0.95$ and $P(p \mid f) = 0.8$.

(1) We want to calculate $P(p)$.

$$P(p) = P(p \mid f)P(f) + P(p \mid f^c)P(f^c) = 0.8 \cdot 0.01 + 0.05 \cdot 0.99 = 0.0575$$

(2) We want to calculate $P(n \cap f)$

$$P(n \cap f) = P(n \mid f)P(f) = 0.2 \cdot 0.01 = 0.002$$

(3) We want to calculate $P(f^c \mid p)$.

$$P(f^c \mid p) = \frac{P(p \cap f^c)}{P(p)} = \frac{P(p \mid f^c)P(f^c)}{P(p)} = \frac{0.05 \cdot 0.99}{0.0575} = 0.8609$$

(4) Frequency since we are using actual data that isn't dependent on opinion.

□

Exercise 30 You and your nine friends are playing truth or dare! In order to decide who starts you decide to take 10 strips of paper of equal length and cut one of them shorter. The first person to pick the shorter one gets to start. Is this a fair way of choosing who begins? (In other words, does everyone have the same probability of choosing the short strip of paper?)
Answer. Yes! The first person has probability:

$$P(1) = \frac{1}{10}$$

The second person has probability:

$$P(2) = \frac{9}{10} \cdot \frac{1}{9}$$

where the $\frac{9}{10}$ represents the first person not getting a short stick and the $\frac{1}{9}$ represents the chance of getting the short strip from the remaining 9.

For the third person we have:

$$P(3) = \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10}$$

Continuing in this way, you can see every person has a $\frac{1}{10}$ chance of pulling the short strip. □