Week 1

Exercise 1 Suppose we have a bag where there are twice as many green marbles as red marbles. What is the proportion of green marbles as:

(1) a fraction

(2) a decimal

(3) a percentage

Answer. We have $2x$ green marbles and x red marbles. So we have $3x$ marbles in total. Therefore $|\Omega| = 3x$ and $|A| = 2x$.

(1) $\frac{2}{3}$

(2) $0.6666\ldots$

(3) $66.666\ldots\%$

□

Exercise 2 Suppose we have the following:

How much wood could a woodchuck chuck if a wood chuck could chuck wood?

(Count each word separately even if it's the same word) What is the probability of the following events:

(1) A word has three letters.

(2) A word contains exactly two vowels.

(3) A word contains an "a" or an "i".

Answer. There are 14 words in total so $|\Omega| = 14$. Note that words are allowed to repeat!

(1) The only word with three letters is "How" and therefore the probability is $\frac{1}{14}$.

(2) The words with exactly two vowels are "wood", "could", "wood", "could", "wood". Therefore the probability is $\frac{5}{14}$.

(3) The words that contain an "a" are: "a" and "a". The words that contain an "i" are "if". Remembering that "or" means we combine the two sets gives us 3 total words. Therefore the probability is $\frac{3}{14}$.

□

Exercise 3 Suppose there's a roulette table with 38 numbers: $1, 2, \ldots, 36, 0, 00$. The numbers $1 - 36$ are either red or black (half/half) and the 0 and 00 are both green. Suppose I bet on red and you bet on black. Once the bets are placed, the ball rolls and lands on a number.

(1) What is the probability that we both lose?

(2) What is the probability that we both win?

(3) What is the probability that at least one of us wins?

(4) What is the probability that at least one of us loses?

Answer. Since there are 38 numbers, our sample space has 38 elements. We will let A be the event "lands on red" and B be the event "lands on black".

(1) This question is really asking for the event "Not red and not black". In other words, $(A \cup B)^c$. Since $A \cup B$ is the set of all numbers $1 - 36$, $(A \cup B)^c = \{0, 00\}$. Therefore the probability is given by $\frac{2}{38}$.

(2) The question is really asking for the event "Red and black". This is represented by $A \cap B$. But since no number red and black, we know $A \cap B = \emptyset$. Therefore the probability is $\frac{0}{38}$.

(3) The question is asking that either A or B to happen, *i.e.*, $A \cup B$. Therefore we have $\frac{36}{38}$.

(4) Notice that the opposite of "one of us loses" is "both of us win", which we calculated earlier. So this is just $(A \cap B)^c$. Since $A \cap B = \emptyset$ we know $(A \cap B)^c = \Omega$ and therefore we have a certain event, *i.e.*, probability is 1.

□

Exercise 4 Suppose we have a box with n tickets in it, each numbered from 1 to n (so that all numbers appear exactly once). If I draw a ticket at random, put it back in the box and pull out another random ticket again, what are the probabilities of the following events:

(1) The same number is pulled out twice.

(2) The two numbers are consecutive integers (*i.e.*, 3 gets pulled and then a 4 or 4 gets pulled and then a 3).

(3) The first number pulled is (strictly) less than the second number pulled.

If instead I draw a random ticket, leave it outside the box and pull out another random ticket, what are the probabilities of the same three events as above.
Answer. For the first part, since we are pulling a number out of n options twice, there are n^2 elements in our sample space. These can be represented by ordered pairs: $(1, 1)$, $(1, 2)$, $\ldots$, $(1, n) \cdots (n, 1)$, $\ldots$, (n, n).

(1) The same number is pulled twice whenever we have (i, i). In other words there are n total possible ways. Therefore our probability is given by $\frac{n}{n^2}$.

(2) The two numbers are consecutive if either $(i, i+1)$ or $(i, i-1)$. In the first case, i can be any number from 1 to $n-1$. In the second case, i can be any number from 2 to n. Taking the union, we have that the probability is $\frac{(n-1)+(n-1)}{n^2} = \frac{2n-2}{n^2}$

(3) The first number pulled is less than the second number if the pair (i, j) is such that $i < j$. When $i = 1$ there are $n-1$ options. For $i = 2$ there are $n-2$ options. Continuing down, we see there are $(n-1) + (n-2) + \ldots + 1$ options in total. This gives

$$\sum_{i=1}^{n-1} i = \frac{(n)(n-1)}{2}$$

So we have that the probability is $\frac{n^2-n}{2n^2} = \frac{n(n-1)}{2n^2}$.

For the second part, after drawing out a number, we leave the number to the side and pull out a second number. That means we are removing all pairs (i, i) from our sample space. Therefore we have that the sample space now has $n(n-1)$ elements.

(1) Since all of the (i, i) were removed, this is an impossible event! There is no way to pull out the same number twice. Therefore our probability is given by $\frac{0}{n(n-1)} = 0$.

(2) In this case, our subset stays the same. So we have $\frac{2(n-1)}{n(n-1)} = \frac{2}{n}$.

(3) In this case, our subset stays the same as well, so we get: $\frac{n(n-1)}{2n(n-1)} = \frac{1}{2}$

□

Exercise 5 Suppose I randomly shuffle a deck of 52 cards. I give you the first card and I take the second card. What are the probabilities of the following events:

(1) You received a 2.

(2) I received a 2.

(3) We both received a 2.

Answer. We first let Ω be the set of ordered pairs (c_1, c_2) where c_1 is the card you received and c_2 is the card I received. There are 52 options for c_1 and 51 options for c_2 (since it can't be the same as c_1). Let A be the event "You received a 2" and B be the event "I received a 2".

(1) We know that A is the subset of Ω whose c_1 is either the two of spades, hearts, diamonds or clovers. Since there are 4 possibilities out of 52 we know the probability that you received a 2 is $\frac{4}{52} = \frac{1}{13}$.

(2) We know that B is the subset of Ω whose c_2 is either the two of spades, hearts, diamonds or clovers. Since there are 4 possibilities out of 52 we know that the probability that I received a 2 is $\frac{4}{52} = \frac{1}{13}$.

Why does it not depend on c_1? It does, but the dependency disappears. We have two options: Either the first card (your card) was a two or it wasn't. If you pulled a two, then my chances of getting a two are:

$$\frac{4}{52} \cdot \frac{3}{51}$$

If you didn't pull a two, then my chances of getting a two are:

$$\frac{48}{52} \cdot \frac{4}{51}$$

To get the total probability, we add these together:

$$\frac{4 \cdot 3}{52 \cdot 51} + \frac{48 \cdot 4}{52 \cdot 51} = \frac{4 \cdot 51}{52 \cdot 51} = \frac{4}{52}$$

(3) If we both received a 2, we're asking for $A \cap B$. This only happens if c_1 and c_2 are both 2. There are 4 cards that are 2. If we deal these out that gives you 4 possibilities and me 3 possibilities (since you have one) therefore we have $4 \cdot 3 = 12$ total options. Therefore the probability is $\frac{12}{52 \cdot 51} = \frac{1}{221}$

□

Exercise 6 Suppose I roll two 4-sided dice. Find the probabilities of the following events:

(1) The maximum of the two numbers rolled is less than or equal to 2.

(2) the maximum of the two numbers rolled is less than or equal to 3.

(3) The maximum of the two numbers equals exactly to 1.

(4) Repeat the previous question for 2, 3, and 4.

(5) What is the sum of the probabilities of the last two questions? Does that make sense?

Answer. Our sample space is going to have 16 different elements of the form (i, j).

(1) This question is asking when $\max(i, j) \leq 2$. This happens when: $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ are rolled. So this happens $\frac{4}{16} = \frac{1}{4}$ of the times.

(2) This question is asking when $\max(i, j) \leq 3$. This happens when we have one of the four from above, but also when

$$(1, 3),\ (2, 3),\ (3, 3),\ (3, 2),\ (3, 1)$$

giving us five more options. So in total we have 9 potential options and our probability is $\frac{9}{16}$.

(3) We now ask when $\max(i, j) = 1$. This only happens when both i and j are equal to 1. Therefore we have $\frac{1}{16}$.

(4) If $\max(i, j) = x$ for some x, then we know either $i = x$ or $j = x$ (or both). For $x = 2$, there are 3 options as we saw above. For $x = 3$ there are 5 options and for $x = 4$ there are 7 options. So our probabilities are respectively:

$$\frac{3}{16}, \frac{5}{16}, \frac{7}{16}$$

(5) Inherently, the sum of the probabilities should equal to 1 since there are no other options. Let's verify this occurs:

$$\frac{1}{16}+\frac{3}{16}+\frac{5}{16}+\frac{7}{16}=\frac{1+3+5+7}{16}=\frac{16}{16}=1$$

□

Exercise 7 Say we're watching a horse race and the presenter says that "my little pony" has a 3 : 1 odd in favour of winning. What is the probability that they will win?

Answer. Since they have 3 : 1 odds in *favour* of winning that means 3 times they will win and 1 time they will lose. So there are 4 options altogether giving a probability of $\frac{3}{4}$ of winning. □

Exercise 8 Say that our best friend is about to have a baby and you know that statistically roughly out of every 60 babies born, roughly 30 are assigned male at birth, 29 are assigned female at birth and 1 are assigned intersex at birth.

(1) What is the sample space?

(2) What is the probability that your best friend's baby is assigned female at birth?

(3) Is this a frequency or a subjective interpretation of probability?

Answer. (1) $\Omega = \{male, female, intersex\}$

(2) There is a 29 in 60 chance that they will be assigned female at birth. So we would say the probability is $\frac{29}{60}$.

(3) This is a frequency interpretation.

□

Exercise 9 Prove that $\binom{n}{k}=\binom{n}{n-k}$

Answer.

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n!}{(n-(n-k))!(n-k)!}=\binom{n}{n-k}$$

□

Exercise 10 Prove that $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$

Answer.

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\frac{(n-1)!}{(k-1)!(n-1-(k-1))!}+\frac{(n-1)!}{k!(n-1-k)!}=\frac{(n-1)!(k+(n-k))}{k!(n-k)!}=\binom{n}{k}$$

□

Exercise 11 Given a normal 52 card deck.

(1) How many different 5 card hands are there?

(2) How many different 5 card hands do not have the number 2?

(3) How many different 5 card hands contain exactly one number 2?

(4) How many different 5 card hands contain all cards of the same suit?

Answer. (1) The first question is asking for the number of combinations of 5 out of 52 elements. This is:

$$\binom{52}{5}=2,598,960$$

(2) This question is asking what happens if we don't have the number 2. In other words the number of combinations of 5 out of 48 elements.

$$\binom{48}{5}=1,712,304$$

(3) If we have exactly one number 2 we know that for the 2 we choose one card out of 4 and for the other ones we choose three cards out of 48. So we get:

$$\binom{48}{4}\binom{4}{1} = 778,320$$

(4) If we want them all to be the same suit we first have to choose a suit giving us 4 different options. Then within each suit we need to choose five cards of the 13 possibilities. This gives:

$$4 \cdot \binom{13}{5} = 5,148$$

□

Exercise 12 Suppose it's your birthday and your two best friends bought you a cake. You're about to cut it when your mom in the other room yells at you to save her a slice too! So you cut the cake into four different slices, one for each person. Obviously since it's your birthday you get the biggest slice which turns out to be the same size as both of your best friends' slices combined (your best friends get the same size slice)! The fourth slice for your mom turns out to be half the size of one of your best friend's slices. How much of the cake did you get?
Answer. In this case it's best to pretend that the cake was sliced multiple times and count up how many slices exist in total. Since your mom has the smallest slice, pretend that all slices are the same size as your mom's. Both of your best friends have slices two times bigger than your mom's slice. So we can consider this as being 2 slices each, giving 5 slices altogether. We also know that your slice is as big as both of your friend's slices combined. That means you would get 4 slices $(2 + 2)$. That means there are 9 slices in total and you get $\frac{4}{9}$ of the cake. □

Exercise 13 Suppose you and your friend are at a fair and you decide to join the raffle. You buy 4 tickets (numbered $71 - 74$) and your friend buys 5 tickets (numbered $89 - 93$). You know that there are 100 tickets in the raffle in total and the tickets are labelled (in order) starting from 1.

(1) What is the sample space?

(2) What subset represents the event "You won the raffle"?

(3) What subset represents the event "You or your friend won the raffle"?

(4) What subset represents the event "You and your friend won the raffle"?

(5) What subset represents the event "The number drawn was 1 away from either you or your friend's tickets, but not exact with any ticket."?

Answer. (1) $\Omega = \{1, 2, \ldots, 100\}$

(2) $A = \{71, 72, 73, 74\}$

(3) $A \cup \{89, 90, 91, 92, 93\}$

(4) $\emptyset$

(5) $\{70, 75, 88, 94\}$

□

Example 14 Let $\Omega = \{0, 1, 2\}$ be the sample space for flipping a coin twice where each outcome shows how many heads you got. Which of the following is an event in this space? If it's an event, give its subset representation and if it's not an event, explain why.

(1) The coin lands on heads twice.

(2) The coin lands on heads at least once.

(3) The coin lands on heads and then lands on tails.

(4) The coin lands on heads once and lands on tails once.

Extra: Why is this question ok when it looks like one of the common student mistakes found in class?
Answer. (1) $\{2\}$.

(2) $\{1, 2\}$.

(3) Impossible since this model doesn't represent order of flipping.

(4) $\{1\}$.

Extra: This is ok because we are being *told* what the sample space is. If we had not been told about the sample space and had just been told to find the sample space of flipping a coin twice randomly, then this would not be the sample space we got. In essence, by being told what the sample space is, we know that these coins are *not* fair coins like we normally consider in class. □

Exercise 15 Suppose we have the following:

How much wood could a woodchuck chuck if a wood chuck could chuck wood?

(1) What is the distribution of the length of the words?

(2) What is the distribution of the number of vowels per word?

(Count each word separately even if it's the same word)
Answer. Recall that we have 14 words in total.

(1) There are 2 words of length 1, 1 word of length 2, 1 word of length 3, 4 words of length 4, 5 words of length 5 and 1 word of length 9. So we have the distribution:

$$P(1) = \frac{2}{14},\ P(2) = \frac{1}{14},\ P(3) = \frac{1}{14},\ P(4) = \frac{4}{14}$$
$$P(5) = \frac{5}{14},\ P(6) = P(7) = P(8) = 0,\ P(9) = \frac{1}{14}$$

(2) There are 8 words with just 1 vowel, 5 words with 2 vowels and 1 word with 3 vowels. So we have the distribution:

$$P(1) = \frac{8}{14},\ P(2) = \frac{5}{14},\ P(3) = \frac{1}{14}$$

□

Exercise 16 Show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Answer.

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup (B \cup C)) \qquad \text{(Add parenthesis)} \\
&= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \qquad \text{(Inclusion-exclusion)} \\
&= P(A) + (P(B) + P(C) - P(B \cap C)) - P((A \cap B) \cup (A \cap C)) \quad \text{(Distribution)} \\
&= P(A) + P(B) + P(C) - P(B \cap C) - (P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))) \quad \text{(Inc-exc)} \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}$$

□

Exercise 17 Suppose we have the events A, B, and Z where

$$P(A) = 0.8 \quad P(B) = 0.4 \quad P(Z) = 0.3$$
$$P(A \cap B) = 0.2 \quad P(A \cap Z) = 0.2 \quad P(B \cap Z) = 0.1$$
$$P(A \cap B \cap Z) = 0$$

Find the following probabilities:

(1) $P(A \cup B)$

(2) $P(A \cup B \cup Z)$

(3) $P(A \cap B^c \cap Z^c)$

Answer. (1)

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \qquad \text{(inclusion-exclusion)} \\ &= 0.8 + 0.4 - 0.2 \qquad \text{(Plug-in)} \\ &= 1 \end{aligned}$$

(2)

$$\begin{aligned} P(A \cup B \cup Z) &= P(A) + P(B) + P(Z) - P(A \cap B) - P(A \cap Z) - P(B \cap Z) + P(A \cap B \cap Z) \quad \text{(Exercise 16)} \\ &= 0.8 + 0.4 + 0.3 - 0.2 - 0.2 - 0.1 + 0 \qquad \text{(Plug-in)} \\ &= 1 \end{aligned}$$

(3)

$$\begin{aligned} P(A \cap B^c \cap Z^c) &= P(A \cap B^c) - P(A \cap B^c \cap Z) \qquad \text{(Difference rule)} \\ &= P(A) - P(A \cap B) - (P(A \cap Z) - P(A \cap B \cap Z)) \qquad \text{(Difference rule)} \\ &= 0.8 - 0.2 - (0.2 + 0) \qquad \text{(Plug-in)} \\ &= 0.4 \end{aligned}$$

□