HW 4

Question 1 You have two coins in front of you: red and blue. The blue coin flips a head with probability $\frac{1}{9}$ and the red coin flips a head with probability $\frac{1}{5}$.

- If both coins flip heads, you go north 3 blocks.
- If blue is heads and red is tails, you go north 1 block.
- If blue is tails and red is heads, you go east 1 block.
- If both coins flip tails, you go east 2 blocks.

What is your expected location after 258 flip? What is the standard deviation for going north in exactly 1 flip of both coins?

Assume coin flips are independent for each coin and that both coins are independent. You must use random variables.

1st lets get distribution for coin tosses:

$P(\text{blue heads, red heads}) = P_{HH} = \frac{1}{9}\cdot\frac{1}{5} = \frac{1}{45} = 0.\overline{22}$

$P(\text{" heads, " tails}) = P_{HT} = \frac{1}{9}\cdot\frac{4}{5} = \frac{4}{45}$

$P(\text{" tails, " heads}) = P_{TH} = \frac{8}{9}\cdot\frac{1}{5} = \frac{8}{45}$

$P(\text{" tails, " tails}) = P_{TT} = \frac{8}{9}\cdot\frac{4}{5} = \frac{32}{45}$

Let N_i be the # of blocks we go North on i^{th} flip.
Let E_i " ——— " East on i^{th} flip.
Let $\mathcal{N}$ be the # of blocks we go north
Let $\mathcal{E}$ be the # of " ——— " east

$\mathcal{N} = N_1 + N_2 + \cdots + N_{258}$

$\mathcal{E} = E_1 + E_2 + \cdots + E_{258}$

$E(X_1+X_2) = E(X_1) + E(X_2)$ "By linearity"

$E(\mathcal{N}) \underset{\text{by linearity}}{=} \sum E(N_i) \underset{\text{by indepe (⇒ Equal distribution)}}{=} 258\, E(N_i)$

$E(\mathcal{E}) = \sum E(E_i) = 258\, E(E_i)$

(N, E)
x: 0 blocks, 1 block, 3 blocks

• $E(N_i) = \sum_{\text{all } x} x\, P(N_i = x)$

$= 3\cdot P_{HH} + 1\cdot P_{HT} + 0\cdot(P_{TH}+P_{TT})$

$= 3\cdot\frac{1}{45} + 1\cdot\frac{4}{45} = \frac{7}{45}$

$E(\mathcal{N}) = 258\, E(N_i) = 258\cdot\frac{7}{45} = \frac{602}{15} \approx 40.\overline{333}$

• $E(E_i) = \underbrace{\sum_{\text{all } x} x\, P(E_i = x)}$

$= 2\cdot P_{TT} + 1\cdot P_{TH} + 0\cdot(P_{HH}+P_{HT})$

$= 8/5 \approx 1.6$

$E(\mathcal{E}) = 258\cdot 1.6 = 412.8$

∴ Expected location: $(40.\overline{33},\ 412.8)$
"North" "East"

$(40, 412)$ ← mode

• $E(N_i^2) = 3^2\cdot P_{HH} + 1^2\cdot P_{HT} + 0\cdot —$

$= \frac{9}{45} + \frac{4}{45} = \frac{13}{45} \approx 0.2\overline{888}$

$SD(N_i) = \sqrt{E(N_i^2) - E(N_i)^2} = \sqrt{\frac{13}{45} - \left(\frac{7}{45}\right)^2} = \boxed{0.5144816}$

Any Random Variable X

$SD(N_i) = \sqrt{E(N_i^2) - E(N_i)^2}$

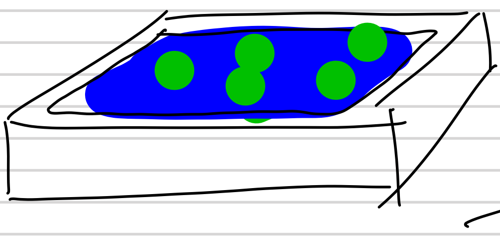
$\hookrightarrow E(N_i^2) = \sum x^{2} P(N_i = x)$

$E(\mathcal{N}^2) = E(\underbrace{(N_1 + N_2 + \cdots N_{258})^2})$

Random Scatters

§3.5 Example 3

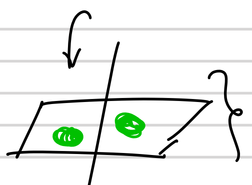
Example 38.2



2000 bacteria in 1000 drops $\to$ 2 bacteria/drop

Expected Value

#1

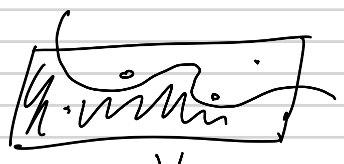


$\to$ We expect ② bacteria on whole dish $\to$ Poisson distribut^{n} $\boxed{\mu=2}$
① " $\to$ " half dish $\to$ " $\mu=1$

#2

$\boxed{\exists \text{ prob } p \text{ that each bacteria will die. What's the distribution}}$

Take an $\boxed{\text{area } B}$ having $\boxed{\text{~~one~~ bacteria}}$ (2)



1/2

$$\boxed{\underbrace{P(1 \text{ Bacterium in } B)}_{=1} \approx 2 \cdot \underbrace{\text{area}(B)}_{2 \cdot \frac{1}{2}}}$$

$$\boxed{P(2 \text{ Bacterium in } B) \approx 0}$$

$P(1 \text{ colony in } B) = P(1 \text{ bacterium in } B \text{ \& it becomes a colony})$
$+ P(2 \text{ bacterium in } B \text{ \& it becomes a colony})$
$= 2 \cdot \text{area}(B) \cdot (1-p) + 0 \cdot (1-p)$

define one

Ok, but let's look at the whole petrie dish.

$$2 \cdot \underbrace{\text{area}(\text{petrie dish})}_{=1}(1-p)$$

$2(1-p)$ } intensity in the random scatter

$\boxed{\text{The Total \# of Colonies has Poisson distribution w/ } 2(1-p) \text{ on whole petrie dish.}}$

parameter.

$p=0$ (aka no death) $\to$ $2(1-0)=2$

$p=1/2$ (aka 1/2 die) $\to$ $2(1-\frac{1}{2}) = 2\frac{1}{2} = 1$

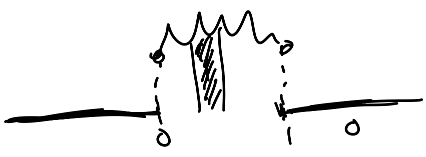
$2 \cdot \frac{3}{4} = \frac{3}{2} = \mu \quad E(X)=\mu$

Exercise 75 Suppose that X has probability density function $f(x) = cx^2(1-x)^2$ for $0 \leq x \leq 1$ and is equal to 0 everywhere else.

(1) What is the value of c?

(2) What is the expected value of X?

(3) What is the variance of X?



$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(X) = \sum x \, P(X=x)$$

X

x

Fundamental Thm of calc

$$f(x)\Big|_a^b = f(b) - f(a)$$

① $$P(-\infty \leq X \leq \infty) = 1$$

$$= \boxed{\int_{-\infty}^{\infty} f(x)\,dx}$$

$$= \int_0^1 c x^2 (1-x)^2\,dx$$

$$= c\int_0^1 x^2(1-2x+x^2)\,dx$$

$$= c\int_0^1 x^2 - 2x^3 + x^4\,dx$$

$$= c\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1 = c\left[\left(\frac{1^3}{3} - \frac{2\cdot 1^4}{4} + \frac{1^5}{5}\right) - \left(\frac{0^3}{3} - \frac{2\cdot 0^4}{4} + \frac{0^5}{5}\right)\right]$$

$$= c\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = c\left(\frac{1}{30}\right) = 1$$

$$\boxed{c = 30}$$

② $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 30x^2(1-x)^2\,dx$$

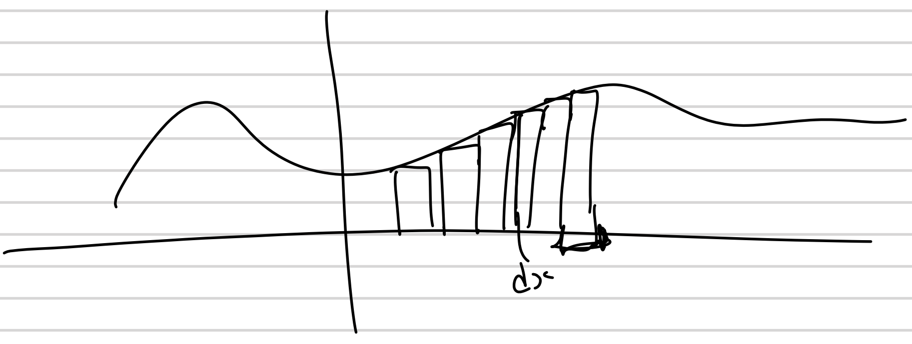
$$= \boxed{1/2}$$

③ $$Var(X) = E(X^2) - E(X)^2 = \boxed{\frac{1}{28}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot 30x^2(1-x)^2\,dx = 2/7$$

$$= \int_0^1 x^2 \cdot 30x^2(1-2x+x^2)\,dx$$

$$= \int_0^1 30 \cdot x^4 - 2x^5 + x^6\,dx$$



Normal

$$P(a \leq X \leq b) = \Phi(b) - \Phi(a)$$

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = \Phi(b) - \Phi(a)$$

fundamental theorem of calculus

Exercise 74 Instead of Chernobyl, pretend we're in a laboratory where we're looking into radioactive substances (safely). We know that radioactive substances release particles known as α-particles. We set-up a counter (like a Geiger counter) to see how many α-particles are given off in a time period. Suppose that we have two different substances and they are emitting α-particles independently of one another. The first substance has a Poisson distribution with parameter 3.87 and the second substance has a Poisson distribution with parameter 5.41 (in a given time frame). What is the probability that the counter is hit by at most 4 α-particles (based on the time frames given by the distributions)?

Exercise 71 You've been hired by Aram to make sure his exercises don't have errors. On average there is 1 error per page. What is the probability that after Aram has written all 300 pages of his exercise sheets there will be at least one page that has at least 5 errors on it? (Assume Poisson distribution and that each page having errors is independent.)

Exercise 70 Suppose we're making chocolate chip cookies and we want the probability of a cookie containing at least 1 chocolate chip to be at least 99%. How many chocolate chips must a cookie contain on average?