Discrete moments

$fn: \{ f(x) = x^n \}$

$$E(X) = \sum_{\text{all } x} x \, P(X = x)$$

$$E(f(X)) = \sum_{\text{all } x} f(x) \, P(X = x)$$

$$P(f(X) = f(x)) \Rightarrow$$

$$P(f^{-1}f(X) = f^{-1}f(x)) \Rightarrow$$

$$P(X = x)$$

$$E(X^2) = \sum_{\text{all } x} x^2 \, P(X = x)$$

$$E(X^{77}) = \sum_{\text{all } x} x^{77} \, P(X = x)$$

Exercise 68 Suppose that we have two independent geometric random variables X_1 and X_2. Let X_1 have parameter p_1 and X_2 have parameter p_2. Find the following:

(1) $P(X_1 = X_2)$

(2) $P(X_1 < X_2)$

(For the second one, you can leave it in a summation.)

(1) $P(X_1 = X_2) = \sum_{k=1}^{\infty} P(X_1 = k, X_2 = k)$

independent

$= \sum_{k=1}^{\infty} P(X_1 = k) \cdot P(X_2 = k)$

inclass

$= \sum_{k=1}^{\infty} (1-p_1)^{k-1} \cdot p_1 \cdot (1-p_2)^{k-1} \cdot p_2$

$= p_1 p_2 \boxed{\sum_{k=1}^{\infty} \left((1-p_1)(1-p_2)\right)^{k-1}}$

$= p_1 p_2 \cdot \left(\frac{1}{1-(1-p_1)(1-p_2)} \right)$

$= \frac{p_1 p_2}{1-(1-p_1)(1-p_2)}$

Geometric series

$\sum_{i=1}^{\infty} q^{i-1} = \frac{1}{1-q}$, $|q| < 1$

$(1-p_1)(1-p_2)$, $0 < 1$, $0 < 1$

(2) $P(X_1 < X_2) = \sum_{k=1}^{\infty} P(X_1 = k, X_1 < X_2)$

$= \sum_{k=1}^{\infty} P(X_1 = k, k < X_2)$

$= \sum_{k=1}^{\infty} P(X_1 = k) P(k < X_2)$

$= \boxed{\sum_{k=1}^{\infty} (1-p_1)^{k-1} p_1 (1-p_2)^{k}}$

$= p_1 \boxed{\sum_{k=1}^{\infty} \left((1-p_1)(1-p_2)\right)^{k-1}} (1-p_2)$

$= \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2)}$

$P(k < X_2)$

$= 1 - P(k \geq X_2)$

$= 1 - \left(1 - (1-p_2)^{k}\right)$

$= (1-p_2)^{k}$

Exercise 69 Suppose you're playing volleyball where the first team to get two points more than their opponents is declared the winner. Let G be the total number of games you played where each game is independent of all other games. Assume that your team ends up winning each game with probability p.

(1) What is the probability that the total number of games played is n?

(2) Find the expected value of G.

(3) Find the variance of G.

① $P(G=n) = \{ 0$ if n is odd

WL | WL | LW | LW | WL | WW or LL

$\underbrace{\quad\quad\quad\quad}_{\frac{n-2}{2}}$ $\quad (2p(1-p))^{\frac{n-2}{2}}$

$n/2$ pairs

$\left(\frac{n}{2}-1\right) = \left(\frac{n}{2}-\frac{2}{2}\right) = \frac{n-2}{2}$

$p^2 + (1-p)^2$

win or lose

$$P(G=n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (2p(1-p))^{\frac{n-2}{2}}\left(p^2+(1-p)^2\right) & \text{if } n \text{ is even} \end{cases}$$

② $E(G) = \boxed{E(2X)}$

Let X be the random variable for a team winning a pair

LL $\quad$ WW $\quad$ WL $\leftarrow p(1-p)$

$(1-p)^2 \quad p^2$ $\quad$ LW $\leftarrow p(1-p)$

no one wins

$E(G) = \sum_{\text{all } x} x\, \underbrace{P(X=x)}_{0 \text{ for odd}}$ $\quad$ ∴ odds aren't counted.

$\underline{X \text{ is geometric w/ probability } (p^2+(1-p)^2)}$

Example 36.1

$\boxed{E(X) = \frac{1}{\text{probability}}}$ $\qquad E(X) = \frac{1}{p^2+(1-p)^2}$

$E(G) = E(2X) = 2E(X) = \frac{2}{p^2+(1-p)^2}$

$E(2X) =$
$\sum 2x P(X=x)$
$= 2\sum x P(X=x) = 2E(X)$

③ $Var(G) = Var(2X) = 4Var(X) = 4(SD(X))^2$

$= \frac{8\,p(1-p)}{(p^2+(1-p)^2)^2}$

36.1

$SD(X) = \frac{\sqrt{q}}{p} = \frac{\sqrt{1-(p^2+(1-p)^2)}}{p^2+(1-p)^2}$

(pg 83)

$= 2p(1-p)$

Ex. 36.1 (part)

$$E(X^2) = \sum_{i=1}^{\infty} i^2 q^{i-1} p = p \boxed{\sum_{i=1}^{\infty} i^2 q^{i-1}}$$

$q^0 + q^1 + q^2 + q^3 + \dots$

$$\boxed{\sum_{i=1}^{\infty} q^{i-1} = \frac{1}{1-q}} \qquad \boxed{\sum_{i=1}^{\infty} i q^{i-1} = \frac{1}{(1-q)^2}}$$

$\boxed{q^0 + 2q^1 + 3q^2 + 4q^3 + \dots}$ $\quad 2 \quad 4 \quad 6 \quad 8$

Know this!

$i=1 \quad i=2 \quad i=3 \quad i=4 \quad i=5 \quad i=6$

LHS

$$1 \cdot \sum_{i=1}^{\infty} i^2 q^{i-1} = 1^2 q^0 + 2^2 q^1 + 3^2 q^2 + 4^2 q^3 + 5^2 q^4 + 6^2 q^5 + \dots$$

$$= 1 \cdot q^0 + 4q^1 + 9q^2 + 16q^3 + 25q^4 + 36q^5 \dots$$

$$- q \sum_{i=1}^{\infty} i^2 q^{i-1} = 1q^1 + 4q^2 + 9q^3 + 16q^4 + 25q^5 + \dots$$

$$(1-q) \sum_{i=1}^{\infty} i^2 q^{i-1} = 1q^0 + 3q^1 + 5q^2 + 7q^3 + 9q^4 + 11q^5 + \dots$$

RHS

$$2 \sum_{i=1}^{\infty} i(q)^{i-1} = 2q^0 + 4q^1 + 6q^2 + 8q^3 + \dots$$

$$- \sum_{i=1}^{\infty} q^{i-1} = q^0 + q^1 + q^2 + q^3 + \dots$$

$$2 \sum_{i=1}^{\infty} i(q)^{i-1} - \sum q^{i-1} = 1q^0 + 3q^1 + 5q^2 + 7q^3 + \dots$$

same!!

$$(1-q) \sum_{i=1}^{\infty} i^2 q^{i-1} = 2 \sum_{i=1}^{\infty} i q^{i-1} - \sum_{i=1}^{\infty} q^{i-1}$$

LHS $\qquad$ RHS

$$\frac{(1-q) \sum_{i=1}^{\infty} i^2 q^{i-1}}{(1-q)} = \frac{2 \frac{1}{(1-q)^2} - \frac{1}{(1-q)}}{(1-q)}$$

$$\sum_{i=1}^{\infty} i^2 q^{i-1} = \frac{2}{(1-q)^3} - \frac{1}{(1-q)^2} \qquad 1-q = p$$

$$= \frac{2}{p^3} - \frac{1}{p^2} \cdot \frac{p}{p}$$

$$= \frac{2-p}{p^3}$$

$$E(X^2) = p \sum_{i=1}^{\infty} i^2 q^{i-1} = p \cdot \left(\frac{2-p}{p^3}\right)$$

$$= \frac{2-p}{p^2}$$

$$E(X^2) = \sum_{i=1}^{\infty} i^2 P(X=i)$$

Here X is geometric
$\Rightarrow P(X=x) = q^{x-1} p$

$$= \sum_{i=1}^{\infty} i^2 q^{i-1} p$$