Joint Distributions

Roll a X on a six-sided die,

Flip a Y on a fair coin

$\boxed{P(X)}$

$P(Y)$

$P(\sim\sim \cap \sim\sim)$

~~$P(X \cap Y)$~~

~~$P(1 \cap \text{Heads})$~~ works only w/ sets

$P(\sim\sim)$ needs to be an event must have a verb

$P(X, Y)$

$P(X=1, Y=\text{heads})$ "and"

if independent → $P(X=1, Y=\text{heads}) = \underbrace{P(X=1)}_{1/6}\underbrace{P(Y=\text{heads})}_{\frac{1}{2}} = \frac{1}{12}$

if not independent → $P(X=1, Y=\text{heads}) = P(X=1 \mid Y=\text{heads})\, P(Y=\text{heads})$

joint distribution table for Y_1 and Y_2.

Exercise 50 (a) Let X be the number of heads which appear in 20 random tosses of a fair coin. (b) Let Y be a number which is randomly picked from a hat with the numbers $\{0, 1, 2, \ldots, 20\}$ (independently from X). Let $Z = \max(X, Y)$. Find a formula for $P(Z = k)$ where $0 \le k \le 20$.

Exercise 51 Suppose that we have two dice which are not fair. The number i appears with probability p_i

(a)

$$P(X=k) = \binom{20}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{20-k} = \binom{20}{k}\frac{1}{2^{20}}$$

$$P(X<k) = \sum_{i=0}^{k-1}\binom{20}{i}\frac{1}{2^{20}}$$

(b)

$$P(Y=k) = \frac{1}{21}$$

$$P(Y<k) = \frac{k}{21}$$

$k = 4$ $\quad Y<k$

$\boxed{0, 1, 2, 3}$ $\quad$ k^{th} $\quad$ 1^{st} $\quad$ $k-1$

$$P(Z=k) = P(\max(X,Y)=k)$$

$$= \underbrace{P(X=k, Y<k)}_{\substack{\max(X,Y)=k \\ X=k \\ Y\neq k}} + P(X<k, Y=k) + P(X=k, Y=k)$$

or $\quad$ or

$$= P(X=k)P(Y<k) + P(X<k)P(Y=k) + \boxed{P(X=k)P(Y=k)}$$

$$= \binom{20}{k}\frac{1}{2^{20}}\cdot\frac{k}{21} + \left(\sum_{i=0}^{k-1}\binom{20}{i}\frac{1}{2^{20}}\right)\cdot\frac{1}{21} + \binom{20}{k}\frac{1}{2^{20}}\cdot\frac{1}{21}$$

$$= \binom{20}{k}\frac{k+1}{2^{20}\cdot 21} + \sum_{i=0}^{k-1}\binom{20}{i}\frac{1}{2^{20}\cdot 21}$$

① $= P(X=k)P(Y\le k) + P(X<k)P(Y=k)$

$$= \binom{20}{k}\frac{k}{2^{20}\cdot 21} + \sum_{i=0}^{k}\binom{20}{i}\frac{1}{21\cdot 2^{20}}$$

② $= P(X=k)P(Y<k) + P(X\le k)P(Y=k)$

$$= \frac{1}{2^{20}\cdot 21}\left(\quad \right)$$

Equal vs Equal in Dist,

I have two coins → coin 1 & coin 2

For both coins → $P(H)=\frac{1}{3}$ $P(T)=\frac{2}{3}$

§2.1

Equal in Distribution

Let X be the flip of coin 1

Let Y be the flip of coin 2

Y \ X	H	T	
H	$\frac{1}{3}\cdot\frac{1}{3}$	$\frac{1}{3}\cdot\frac{2}{3}$	1/3
T	$\frac{2}{3}\cdot\frac{1}{3}$	$\frac{2}{3}\cdot\frac{2}{3}$	2/3
	1/3	2/3	

$P(X=x)=P(Y=x)\ \forall x$

$P(X=H)=P(Y=H)=\frac{1}{3}$

$P(X=T)=P(Y=T)=\frac{2}{3}$

Equal

Let X be the flip of coin 1

Let Y be the flip of coin 1

(is Equal in Dist)

Y \ X	H	T	
H	1/3	0	1/3
T	0	2/3	2/3
	1/3	2/3	

+ =1

Equal.

=Equal in Dist

$P(X=H)=P(Y=H)=\frac{1}{3}$

$P(X=T)=P(Y=T)=\frac{2}{3}$

$\boxed{P(X=Y)=1}$ Equal.

	1	2	3	4
1	a	0	0	0
2	0	b	0	0
3	0	0	c	0
4	0	0	0	d

a + b + c + d = 1

		1/3
		2/3
1/2	1/2	

replacement. In each of the cases find the probability $P(X \leq Y)$.

Exercise 49 Let X_1 and X_2 be the numbers obtained by rolling two rolls of a fair die. Let $Y_1 = \max(X_1, X_2)$ and $Y_2 = \min(X_1, X_2)$. We already calculated the joint distribution table for X_1 and X_2 in class. What is the joint distribution table for Y_1 and Y_2?

Exercise 50 Let X be the number of heads which appear in 20 random tosses of a fair coin. Let Y be a

$X_2 \backslash X_1$	$X_1=1$	$X_2=2$	3	4	5	6
1	$\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

(min) $Y_2 \backslash Y_1$ (Max)	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$11/36$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$9/36$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$7/36$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$5/36$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$3/36$
6	0	0	0	0	0	$\frac{1}{36}$ min = max	$1/36$
Totals	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$	1

$\Rightarrow$ Not Equal in Distribution

Exercise 51 Suppose that we have two dice which are not fair. The number i appears with probability p_i for the first die and with probability p'_i for the second die. Let S be the sum of numbers rolled with these two dice. Find the formula for $P(S = 2)$, $P(S = 7)$ and $P(S = 12)$.