Distribut^{n}s

"Count" **Discrete** 

Bernoulli

↳ Something happens or not

$P(X=x) = p$ or $(1-p)$

Binomial → "multiple trials" independent

"random sampling with replacement"

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

"Not countable" **Continuous** 

Normal Distribut^n

$$P(a \le X \le b) = \underbrace{\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz}_{\approx\ \Phi(b) - \Phi(a)}$$

→ Use for approximat^n Binomial when p is close to $1/2$

Random sampling w/o replacement → ~~"Hypergeometric distribution"~~

$$\underbrace{P(X=r, Y=n)}_{p(r \text{ red in } n \text{ pulls})} = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$

N = total populat^n
$R \subseteq N$ = # {Subset of population}
n = smaller population (what we pull out)
r = # of R from smaller population

Poisson distribution

$$P(X=k) = e^{-\mu}\frac{\mu^k}{k!}$$

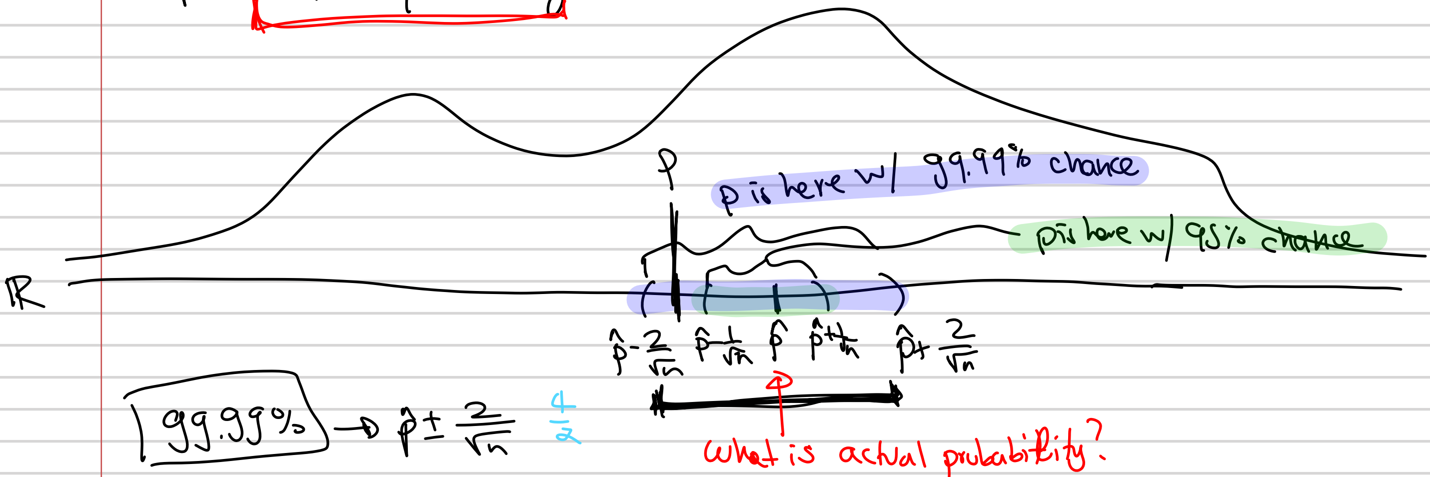
→ Use as approximat^n for Binomial when μ doesn't change a lot (or when p is really small/large)

Confidence Intervals

$\boxed{\hat{p}}$ = real world data

$\downarrow$ as $n \to \infty$

p = actual probability



$\boxed{99.99\%} \to \hat{p} \pm \frac{2}{\sqrt{n}}$ $\quad \frac{4}{2}$

95% $\to \hat{p} \pm \frac{1}{\sqrt{n}}$ $\quad \frac{2}{2}$

$\Phi(-z, z) = \#$

confidence $\hat{p} \pm \frac{z}{2\sqrt{n}}$

Example

heads

$n=9$

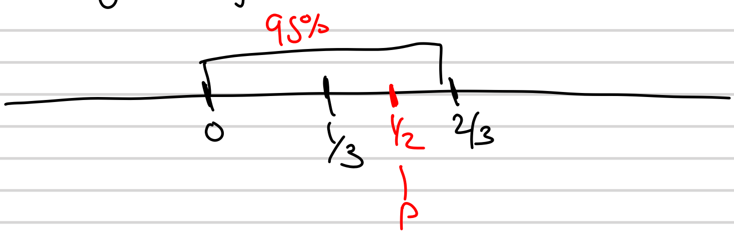
T	H	T	T	T	H	T	H	T
$\frac{0}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{3}{8}$	$\frac{3}{9}$

$\hat{p} = \frac{1}{3}$

$\hat{p} \pm \frac{1}{\sqrt{9}} = \underbrace{\hat{p} \pm \frac{1}{3}}_{\text{95\% confidence interval.}}$

$\boxed{p = 1/2}$

95% conf. int



Exercise 43 A hat contains 1000 random numbers in it. Only two of the numbers are prime, and the rest are composite (not prime).

(1) What is more likely to occur if we pull 1000 numbers from the box with replacement:

- Less than 2 prime numbers are pulled
- Exactly 2 prime numbers are pulled
- More than 2 prime numbers are pulled

(2) If we repeat the 1000 pulls just like the previous question, but this time do it twice in a row (so, do 1000 pulls with replacement; and then do a second 1000 pulls with replacement); what are the chances that we get the same number of prime numbers in both 1000 pulls? (estimate using Poisson)

$$\boxed{p = \frac{2}{1000} = \frac{1}{500}}$$

where X is number of prime #s pulled.

$\mu = np = 1000 \cdot \frac{1}{500} = 2$

- $\boxed{P(X<2)} = P(X=0) + P(X=1)$ Poisson $\left(e^{-\mu}\frac{\mu^k}{k!}\right)$

$$\approx e^{-\mu}\frac{\mu^0}{0!} + e^{-\mu}\frac{\mu^1}{1!}$$
$$= e^{-2} + e^{-2}\cdot 2$$
$$= 3e^{-2}$$
$$= 0.406006$$

- $P(X=2) \approx e^{-\mu}\frac{\mu^2}{2!} = 0.270671$

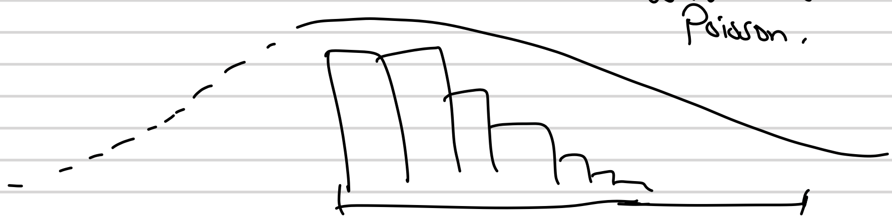
- $P(X>2) = \sum_{i=3}^{1000} P(X=i)$

$$= 1 - P(X=2) - \underbrace{P(X=1) - P(X=0)}$$
$$= 1 - P(X=2) - P(X<2)$$
$$\approx 1 - 0.406006 - 0.270671$$
$$= 0.323323$$

p - smaller

"< 5"

(np or n(1-p)) μ is small want to use Poisson.



$X_1 = $ # of primes in 1^{st} set

$X_2 = $ # of primes in 2^{nd} set

P(Both have same # of prime #s)

$$= \sum_{k=0}^{1{,}000} P(X_1 = k, X_2 = k)$$

$$= \sum_{k=0}^{1000} \underbrace{P(X_1 = k)}\, \underbrace{P(X_2 = k)}$$

$$\approx \sum_{k=0}^{1{,}000} e^{-\mu} \frac{\mu^k}{k!}\, e^{-\mu} \frac{\mu^k}{k!}$$

$$= \sum_{k=0}^{1000} (e^{-\mu})^2 \frac{(\mu^k)^2}{(k!)^2}$$

$\mu = 2$

$$= \sum_{k=0}^{1000} e^{-2\cdot 2} \frac{2^{2k}}{k!\,k!}$$

$$= e^{-4} \left(\sum_{k=0}^{1000} \frac{2^{2k}}{(k!)^2} \right)$$

$$= e^{-4} \left(\frac{2^{2\cdot 0}}{(0!)^2} + \frac{2^{2\cdot 1}}{(1!)^2} + \frac{2^{2\cdot 2}}{(2!)^2} + \cdots \right)$$

$$\approx 0.018315639 \,(1 + 4 + 4 + 1.777.. + 0.4444.. + 0.0711 + 0.00790 + 0.00064 + \cdots)$$

$k=7$, stop.

$$\approx 0.2070$$

$$\sum P(\text{Both have } k \text{ primes})$$

$$= \underbrace{\sum P\left(1^{st} \text{ set has } k \text{ primes } \underline{\text{and}} \;\; 2^{nd} \text{ set has } k \text{ primes}\right)}$$

$$\underbrace{\sum_{k=0}^{1000} \left(\binom{1000}{k} p^k (1-p)^{n-k} \right)^2}$$

Exercise 46 Suppose you go to a small fair with 9 of your friends and you each purchase 10 tickets in the fair's raffle. Later on, you found out, your group of 10 people were the only people to purchase tickets for the raffle! Out of the 100 tickets bought, 3 of them will be chosen randomly as winning tickets.

(1) What are the chances one person will get all 3 winning tickets?

(2) What are the chances that 3 different people will win something?

(3) What are the chances that 2 different people will win?

① 1 person getting all 3 winning tickets

total options:

$$\binom{100}{3} = \frac{100!}{3!\,97!} = \frac{100\cdot 99\cdot 98}{3\cdot 2} = 50\cdot 33\cdot 98$$

Which person gets all 3: $\binom{10}{1} = \frac{10!}{1!\,9!} = 10$

Which 3 tickets win: $\binom{10}{3}$

$$\rightarrow 10\cdot\binom{10}{3} = 10\cdot\frac{10!}{3!\,7!} = 10\cdot\frac{10\cdot 9\cdot 8}{3\cdot 2} = 10\cdot 10\cdot 3\cdot 4$$

$$P(\text{1 person wins}) = \frac{\binom{10}{1}\cdot\binom{10}{3}}{\binom{100}{3}} = \frac{10\cdot 10\cdot 3\cdot 4}{50\cdot 33\cdot 98} = \boxed{0.00742115}$$

② 3 people win (each has 1 winning ticket)

Which 3 people win → $\binom{10}{3}$

Which ticket wins (per person) → $\left(\binom{10}{1}\right)^3$

$$P(\text{3 people win}) = \frac{\binom{10}{3}\left(\binom{10}{1}\right)^3}{\binom{100}{3}} = \frac{10\cdot 10\cdot 10\,\binom{10}{3}}{50\cdot 33\cdot 98} = \frac{10^3\cdot 10\cdot 3\cdot 4}{50\cdot 33\cdot 98}$$

$$\boxed{= 0.742115}$$

③ 2 people win (one wins with 1 tick, one wins with 2)

$$P(\text{2 people win}) = 1 - P(\text{1 person winning}) - P(\text{3 people winning})$$

$$= 1 - 0.00742115 - 0.742115$$

$$= 0.250464$$

Which 2 people win: $\binom{10}{2}$

Winning 1 ticket : $\binom{10}{1}$

Winning 2 tickets : $\binom{10}{2} = \frac{10!}{2!\,8!} = \frac{10\cdot 9}{2\cdot 1} = 5\cdot 9 = 45$

Which of the two gets the two tickets

$\binom{2}{1} = 2$

$$\frac{\binom{2}{1}\binom{10}{2}\binom{10}{2}\binom{10}{1}}{\binom{100}{3}} = \frac{2\cdot 45\cdot 45\cdot 10}{50\cdot 33\cdot 98} = 0.250464$$

1 has 2 ticket

has 1 ticket

$$\cancel{\binom{10}{2}}\binom{2}{1}\binom{10}{2}\binom{10}{1}$$

choose people