Week 6

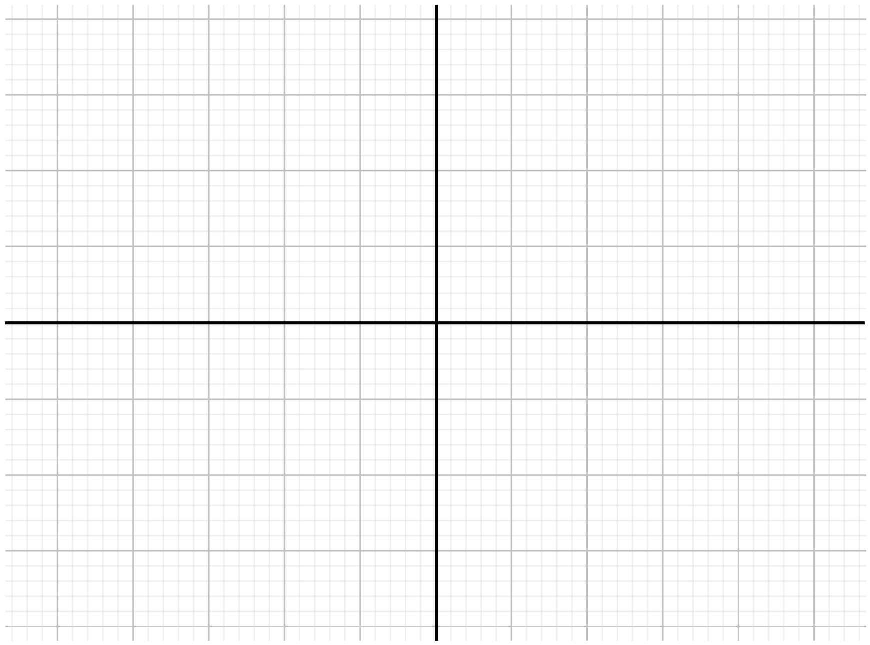
10–14 Feb 2020

6.1 Derivatives of Trig functions - §3.3

Let's try and find the derivative of $\sin(x)$. By definition we have:

$$\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \underline{\hspace{10cm}} \\
&= \underline{\hspace{17cm}} \\
&= \underline{\hspace{15cm}} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}$$

So we need to figure out what these last two limits are to find the derivative. We're going to use the squeeze theorem! First, we want to show that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$ where θ is an angle.



Therefore, intuitively, we know that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$. Now, we take the limits of everything as θ goes to 0.

$$\lim_{\theta \to 0} \cos(\theta) = \underline{\hspace{6cm}}.$$

Therefore, by the squeeze theorem $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = \underline{\hspace{0.7cm}}$.

Now let's look at $\frac{\cos(\theta)-1}{\theta}$.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} &= \underline{\hspace{15cm}} \\
&= \underline{\hspace{10cm}} \\
&= \underline{\hspace{10cm}} \\
&= \underline{\hspace{14cm}} \\
&= -\lim_{\theta \to 0} \frac{\sin(\theta)}{\cos(\theta) + 1} \\
&= \underline{\hspace{4.5cm}} \\
&= \underline{\hspace{0.7cm}}
\end{aligned}$$

Therefore,

$$f'(x) = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \underline{\hspace{4cm}}$$

Using very similar methods, we can prove derivatives for all of the trig functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(x)$	______	$\csc(x)$	______
$\cos(x)$	______	$\sec(x)$	______
$\tan(x)$	______	$\cot(x)$	______

Example 6.1 Let's use the quotient rule to show one of these. Suppose that $f(x) = \frac{\sin(x)}{\cos(x)} = \tan(x)$ and find $f'(x)$.

Exercise 6.2 With a partner, find the derivative of $f(x) = \sin(x)\cos(x)$.

6.2 The chain rule - §3.4

We did addition, subtraction, multiplication and division, but there's one operation we haven't touched yet: composition. What is the derivative of $(f \circ g)$?

In prime notation, we have

This is called the *chain rule*. We won't prove it in class, but it's available in the book if you want to see how it works.

Example 6.3 Let $f(x) = \sqrt{\sin(x)}$. What is $f'(x)$?

Exercise 6.4 Let $f(x) = \frac{1}{\sqrt[3]{x^2+3x}}$. With a partner, find $f'(x)$.

Let's use the chain rule to find the derivative of an exponential function. Let $f(x) = b^x$, and we want to find $f'(x)$. We're going to use the fact that $b = e^{\ln(b)}$. So we have

Example 6.5 Next, we look at an example from chemistry! Suppose we have two substances: substance A and substance B; and when we mix them together we get a substance C. Substances A and B are known as the *reactants* and substance C is known as the *product*. The *concentration* of a reactant or product is normally denoted by the number of moles per liter where a mole is roughly $6.022 \times 10^2 3$ molecules (this number is called *Avogadro's number*). So the concentrations of each substance is given by a function over time. The average concentration of substance C is given by $\frac{\Delta C}{\Delta t}$. As one might expect, the instantaneous concentration at a given time is given by $\frac{dC}{dt}$.

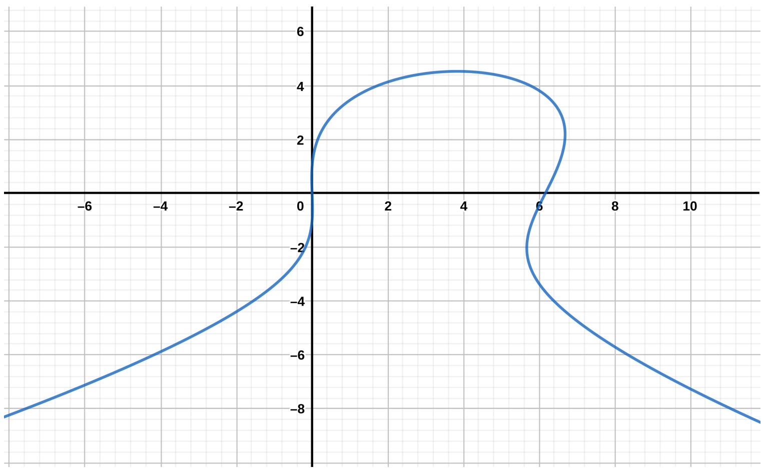
Suppose the concentration of C over a given time is given by the function $c(t) = \cos(2t^2)$. What is the concentration of c at $t = 1$?

So first we need to calculate $c'(t)$.

6.3 Implicit differentiation - §3.5

So far, we've been looking at functions that have been explicitely defined: $f(x) = \cdots$. But what happens if our function is implicitly defined:

$$y^3 + 6x^2 - 2xy = 37x + y$$



Are there ways to find the tangent to the curve? (aka the derivative) _______

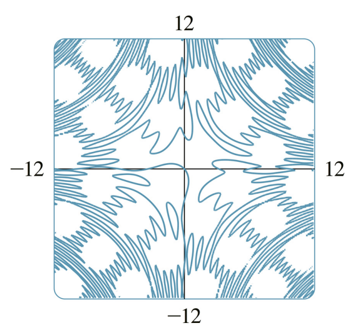
What we do is we break down our differentiation by small sections. This is best done with an example.

Example 6.6 Let $x^3 + y^3 = 37$.

Let's try a little more complicated example.

Example 6.7 Let $y^3 + 6x^2 - 2xy = 37x + y$.

Exercise 6.8 Let $\sin(xy) = \sin(x) + \sin(y)$.



With a partner, find y'.

Example 6.9 Let $x^3 + y^3 = 37$. Find y''.

6.4 Derivatives of inverse trig functions - §3.5

We can use implicit differentiation in order to find the derivatives of inverse trig functions. Recall that $\arcsin(x) = \sin^{-1}(x) = y$ means that $\sin(y) = x$ when $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$. So let's test it out.

$$\begin{aligned} \frac{d}{dx}\sin(y) &= \frac{d}{dx}x \\ \frac{d}{dy}\sin(y)\frac{dy}{dx} &= 1 \\ \cos(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \end{aligned}$$

Since $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ then $\cos(y) \geq 0$. And since $\sin^2(y) + \cos^2(y) = 1$ we have

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \underline{\hspace{4cm}}.$$

In other words

Using similar approaches, we can find the derivatives for every inverse trig function:

$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\text{arccsc}(x)$	$-\frac{1}{x\sqrt{x^2-1}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$\text{arcsec}(x)$	$\frac{1}{x\sqrt{x^2-1}}$
$\arctan(x)$	$\frac{1}{1+x^2}$	$\text{arccot}(x)$	$-\frac{1}{1+x^2}$

6.5 Derivatives of logarithm functions - §3.6

What is the derivative of the function $f(x) = \log_b(x)$?

Recall that $y = \log_b(x)$ means that $b^y = x$. So let's implicit differentiate!

$$\frac{d}{dx}b^y = \frac{d}{dx}x$$

$$\underline{\hspace{3cm}} = \underline{\hspace{0.5cm}}$$

$$\underline{\hspace{3.5cm}} = \underline{\hspace{0.5cm}}$$

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

The derivative comes from Exercise 6.2.

What this means is that if $b = e$ then

Example 6.10 Let $f(x) = \ln(\cos(x))$, what is $f'(x)$?

Example 6.11 Let $f(x) = \left(\ln\left(\frac{x^3}{x^2+1}\right)\right)^2$, what is $f'(x)$?

Exercise 6.12 Let $f(x) = x^2 \ln\left(\frac{x+3}{x^2}\right)$, with a partner, find $f'(x)$.

The logarithm can actually help us differentiate more complicated rational functions. This is best demonstrated with an example.

Example 6.13 Let

$$f(x) = \frac{x^{\frac{9}{4}}(x^3-1)^{\frac{4}{3}}}{(7x^2-1)^4}$$

and find $f'(x)$. This equation is super complicated and we can use the chain rule, the product rule and the quotient rule to solve it, but that's going to get very complicated very fast. Instead, we can use logarithms to simplify everything. We do this in 3 steps.

(1) ____

(2) ____

(3) ________________ .

Example 6.14 Let's try it again with a super simple example. Let $f(x) = x^x$.

(1)

(2)

(3)

Exercise 6.15 Let

$$f(x) = \frac{x^{\frac{4}{3}} \left(x^3 - 3x^2\right)^{\frac{7}{3}}}{\sqrt{x^2+1}}$$

with a partner, find $f'(x)$.