Week 12

23–27 Mar 2020

12.1 Summation - §Appendix E

We're going to kind of change topics. Remember that when we're looking at a graph, we used derivatives to find the slope at any point. We're now going to change our question. How can we calculate the ________ ?

Most of the time, it's easy. For a square we have the length times the height. For a triangle we have one half the base times the height. For a circle we have πr^2. But what if we want to calculate the area under a curve?

Before looking fully into this topic, we're going to take a quick detour into summations.

Example 12.1 Say that we want to add 1 to itself ten times. We can do it like: $1+1+1+1+1+1+1+1+1+1 = 10$. But this becomes more complicated if we want to do more than ten. Say we want to do it 25 times:

$$1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1 = 25.$$

This is way to long and almost imposible to read. So we want to rewrite this in a more visible manner. For adding 1 to itself, it's pretty easy: we do $1 \cdot n$ where n ist he number of times we want to add 1 to itself.

How about if we want to add every integer from 1 to 10: $1+2+3+4+5+6+7+8+9+10 = 55$. What if we want to do from 1 to 25?

$$1{+}2{+}3{+}4{+}5{+}6{+}7{+}8{+}9{+}10{+}11{+}12{+}13{+}14{+}15{+}16{+}17{+}18{+}19{+}20{+}21{+}22{+}23{+}24{+}25 = \underline{\qquad\qquad}$$

This is way to complicated and we want to condense this summation by introducing a new notation ___ .

Example 12.2 Here are the two examples from before.

$$\sum_{i=1}^{10} 1 = 10 \qquad \sum_{i=1}^{10} i = 55$$

In general, we have something like the following

$$\sum_{i=m}^{n} a_i =$$

The i are called the *indices of summation* and they can be thought of as functions.

We can also do this sum without end! If we want to take the summation forever then we let $n = \infty$: $\sum_{i=m}^{\infty} a_i$.

Example 12.3 Let's do some examples.

$$\sum_{i=m}^{n} a_i = \sum_{i=1}^{4} 2i - 1 = 2(1) - 1 + 2(2) - 1 + 2(3) - 1 + 2(4) - 1 = 2 + 4 + 6 + 8 - 4 = 16$$

$$\sum_{i=m}^{n} b_i = \sum_{i=3}^{5} i^2 + 1 = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\sum_{j=x}^{y} i_j = \sum_{j=-4}^{-1} 3 = \underline{\qquad\qquad\qquad\qquad}$$

And in the other direction, we have

$$2 + 2 + 2 + 2 + 2 = \sum_{i=1}^{5} 2$$

$$5 + 8 + 11 + 14 = \underline{\qquad\qquad}$$

$$1 + 0 - 1 - 2 - 3 = \underline{\qquad}$$

Here rae some properties of summations using sigma notation:

Proposition 12.4 *(1)* $\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j$.

(2) $\sum_{i=m}^{n} a_i = \sum_{i=1}^{n-m+1} a_{i+m-1}$.

(3) $\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i$.

(4) *Pour* $c \in \mathbb{R}$, $\sum_{i=m}^{n} c = (n-m+1)c$.

(5) *Pour* $c \in \mathbb{R}$, $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$.

(6) $\sum_{i=1}^{n} (a_i + b_i) = (\sum_{i=1}^{n} a_i) + (\sum_{i=1}^{n} b_i)$

(7) $\sum_{i=1}^{n} (a_i - a_{i-1}) =$ ____________. *telescoping sum*

(8) $\sum_{i=1}^{n} i =$ __________.

(9) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(10) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

(11) $\sum_{i=1}^{n} r^i = \frac{r^{n+1}-r}{r-1}$.

Examples 12.5

$$\sum_{i=1}^{100} i = \frac{100 * (100+1)}{2} = \frac{10100}{2} = 5050$$

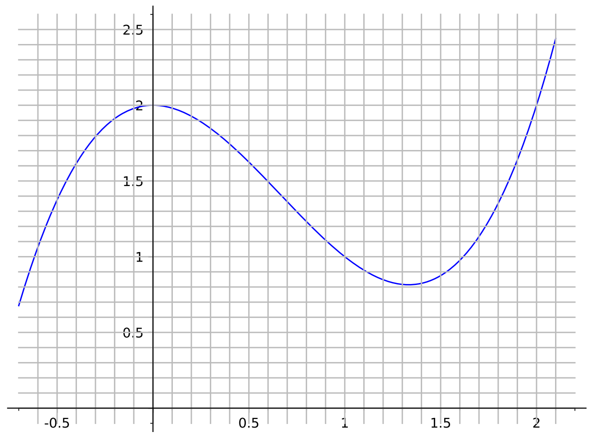
$$\sum_{k=5}^{12} k^2 = \underline{\hspace{10cm}}$$

Exercise 12.6 Try one with a partner.

$$\sum_{i=1}^{20} i^2 - i - 10$$

12.2 Areas and distances - §5.1

Suppose we have the following function: $f(x) = x^3 - 2\,x^2 + 2$:



Since we already kinda know how to look at tangents to a curve, let's look at area under a curve. Say we want to calculate the area under the curve between 0 and 2. How can we do this?

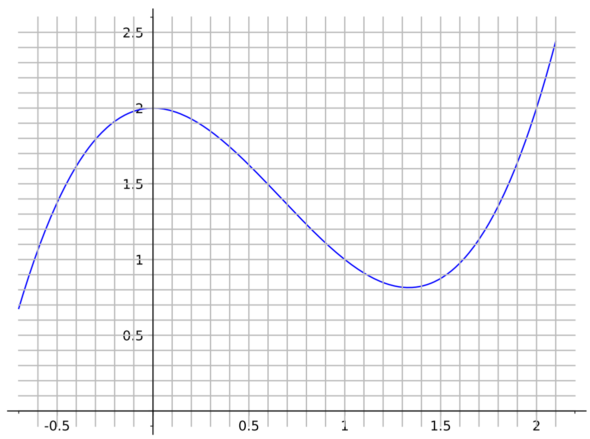
Using our techniques, we can estimate that the area should be: ___.

In essence, we are generally using the following formula for trying to calculate the area:

$$\sum_{i=0}^{n} f(x_i)\Delta x$$

But we can look at this function differently. We can get an estimation by looking at the right-hand sides of our rectangles:

$$\sum_{i=0}^{n} f(x_{i+1})\Delta x$$



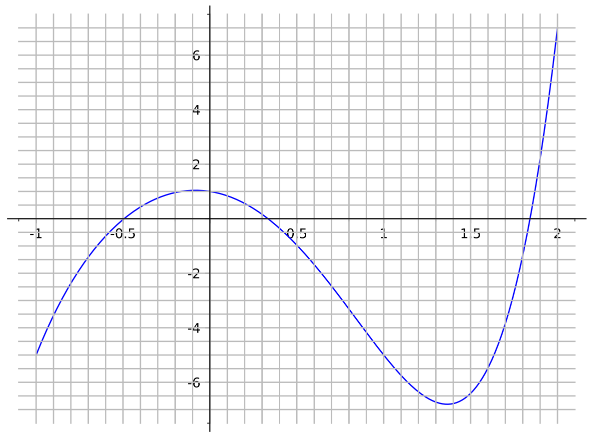
Doing this with our example, we'd get:

$$\begin{aligned}&\sum_{i=0}^{9} f\left(\frac{i+1}{5}\right)\Delta x\\ &\quad= \left(\frac{241}{125}+\frac{218}{125}+\frac{187}{125}+\frac{154}{125}+1+\frac{106}{125}+\frac{103}{125}+\frac{122}{125}+\frac{169}{125}+2\right)\frac{1}{5}\\ &\quad= \frac{67}{25} \approx 2.68000000000000\end{aligned}$$

Notice how these two values are similar to one another!

12.3 Definite intgral - §5.1 – 5.2

Now let's look at a different function, and we're going to be a little more precise in how we are defining these rectangles. We will work with $f(x) = x^5 - 6\,x^2 - x + 1$:



Given a closed interval $[a, b]$, a *partition* of an interval of numbers x_i is such that:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

This gives us little sub-intervals:

$$[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

If the intervals are all the same size ($\frac{b-a}{n}$) we say that the partition is *regular*.

Example 12.7 For some examples, let $a = 2$ and $b = 5$. The following are two different partitions:

$$2 < 2.1 < 3 < 4.2 < 4.5 < 4.51 < 5$$
$$2 < 3 < 4 < 5$$

and they each define different intervals:

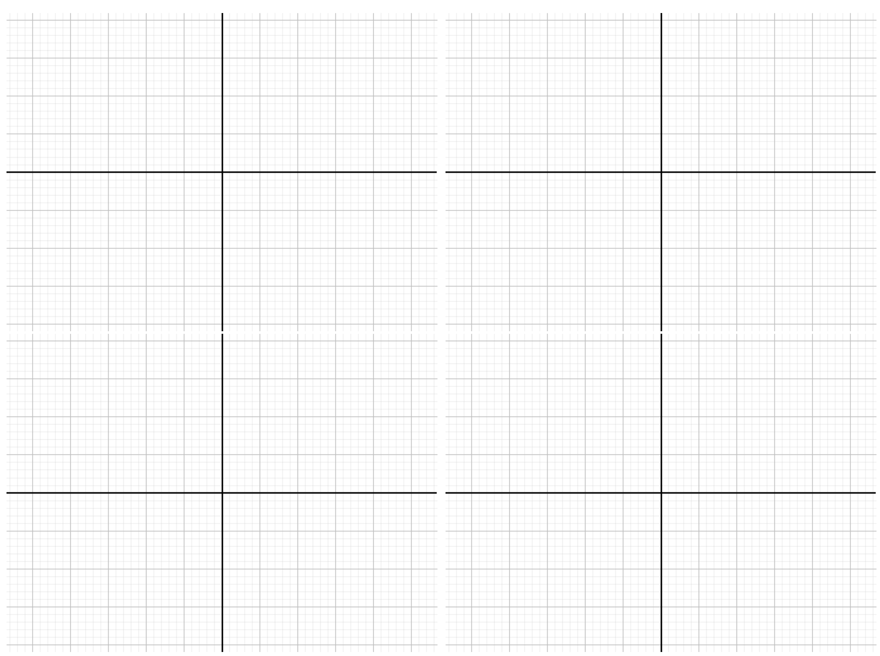
$$[2, 2.1],\ [2.1, 3],\ [3, 4.2], [4.2, 4.5],\ [4.5, 4.51],\ [4.51, 5]$$
$$[2, 3], [3, 4],\ [4, 5]$$

Which partition is regular? __

Exercise 12.8 Given $a = 0$ and $b = 1$, with a partner, give a regular partition:

Always assume regular The *length of an interval* is the difference between the start and end of the interval: $\Delta x_i =$ ________________ for the interval $[x_{i-1}, x_i]$.

We are now ready to define our summations! The *Riemann sum* of a function on an interval $[a, b]$ is the sum: where $x_i^\star \in [x_{i-1}, x_i]$ is some arbitrary number.



We've already seen two of these! The *right-hand sum* is the Riemann sum where we let $x_i^\star = x_i$. The *left-hand sum* is the Riemann sum where we let $x_i^\star = x_{i-1}$. The *midpoint sum* is the Riemann sum where we let $x_i^\star = \frac{x_{i-1}+x_i}{2}$.

But what we reeeeaaaallllly want is to find the *exact* area, not just approximations! What can we do to find the exact area?

We're going to use our favorite tool so far in class: limits. Yes. . . limits.

The limit we're going to use is actually the following limit: If this limit

exists we call say that the function is *Riemann-integrable*.

BUT, this notation is horrible. There are a million and a half symbols. Who wants to write that each time? No one. So we're going to use another notation that (I think) everyone has already seen: the integral.

Remark 12.9 Some history! We use Σ for summation because summation starts with s. So, we need a new symbol for the integral. Luckily, Gottfried Leibniz (German) had came up with the perfect notation back in 1675! He used ancient German's long s: ſ. (Nowadays, German's long s is written as "ß".) Why? Because the integral is a summation of realy small intervals. So ſ is perfect for that since it keeps it as an "s", but is a different s! (Even in English we used to have ſ! It wasn't until between 1800 and 1820 that this letter disappeared from English.)

We changed the "s", but we also need to change the Δ since we're taking the limit. Since we're taking the limit, these intervals are getting smaller and smaller. In other words, it would be nice to represent that by using a "small" Δ. So what's the lower-case of Δ? It's δ of course! But, Leibniz, who was one of the founders of calculus, translated the greek to German and the δ became a d. This is where Leibniz notation: $\frac{dy}{dx}$ actually came from!

So, *if* our function is Riemann-integrable, we will write:

$$\underline{\qquad\qquad\qquad\qquad} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^\star)\Delta x_i.$$

Some definitions:

- The a is the *lower limit of integration*,
- The b is the *upper limit of integration*,
- The $f(x)$ is the *integrand*,
- The x is called the *variable of integration* or the *independent variable*.

Theorem 12.10 *If a function $f(x)$ is ____________________ on the interval $[a, b]$ or if $f(x)$ has only a finite number of jump discontinuities, then the function $f(x)$ is ____ on $[a, b]$.*

Example 12.11 Let's look at an (easy) example. Find $\int_1^2 x^2\, dx$.

That was disgusting. The whole point of the next few days (and the next class) is to try and make this easier.

12.4 Properties of definite integrals - §5.2

Proposition 12.12 *Let $f(x)$ be a continuous function on $[a, b]$ and let c be a constant.*

(1) $\int_a^b f(x)\,dx = \int_a^b f(t)dt = \int_a^b f(u)du$

(2) $\int_a^a f(x)\,dx =$ __

(3) $\int_a^b c\,dx = c(b-a)$

(4) $\int_a^b f(x)\,dx =$ ______________________

(5) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(6) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(7) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$ __________________