Week 10

9–13 Mar 2020

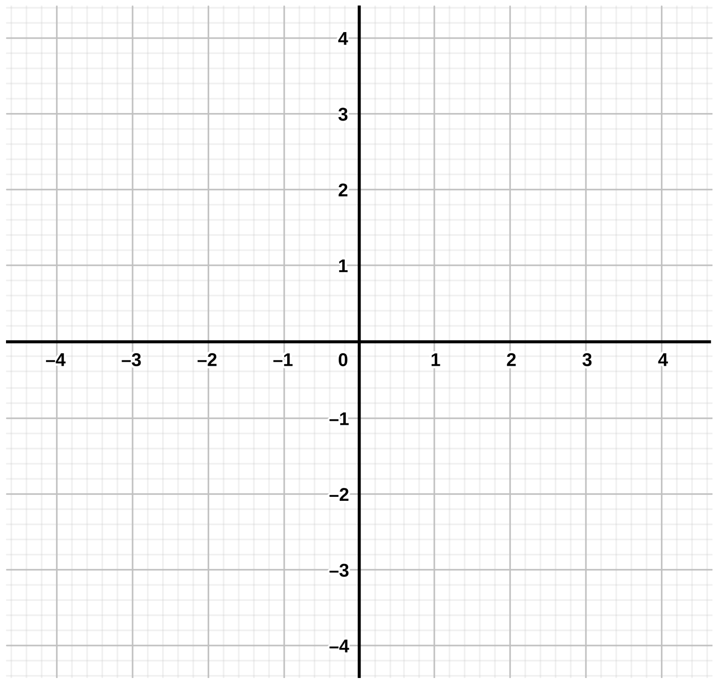
10.1 Derivatives and graphing - §4.3

Our next goal is to look at what the derivative can inform us about the original function. Since the derivative is the slope of the tangent line, when the slope is positive we know the original function is increasing. Likewise, if the slope is negative it's decreasing.

- If $f'(x) > 0$ for all x in an interval, then f is increasing on that interval.
- If $f'(x) < 0$ for all x in an interval, then f is decreasing on that interval.

Example 10.1 Let $f(x) = x^4 + 4x^3 + 4x^2 + 4$. Find which intervals the function is increasing and decreasing.

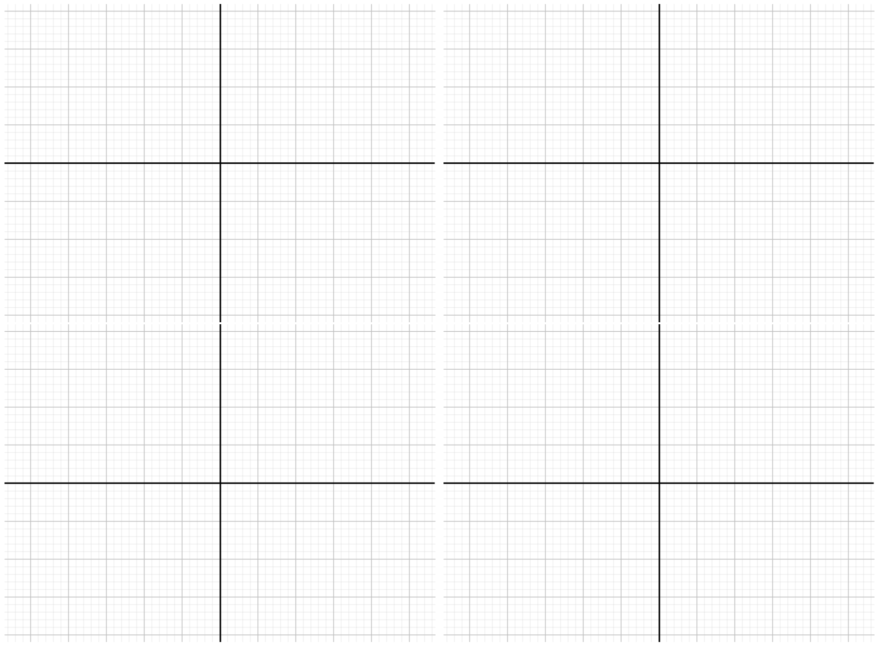
Ok, but how can we use this information to actually draw the graph?



Notice that this tells us a lot about the function! We can summarize this in the following test.

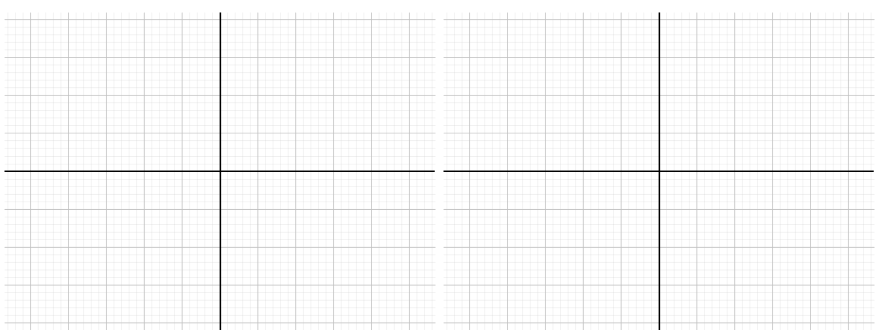
Theorem 10.2 (The first derivative test) *Let c be a critical number of a continuous function f.*

- *If f' changes from positive to negative at c, then f has a local maximum at c.*
- *If f' changes from negative to positive at c, then f has a local minimum at c.*
- *If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.*



Can we say even more if we look at f''? Yes! It turns out the second derivative helps us distinguish "how" a function is increasing/decreasing. In fact, it tells us about its ________________ .

Definition 10.3 If the graph of f lies above all of its tangents on an interval I, then it is called *concave upward* on I. If the graph of f lies below all of its tangents on an interval I, then it is called *concave downward* on I.



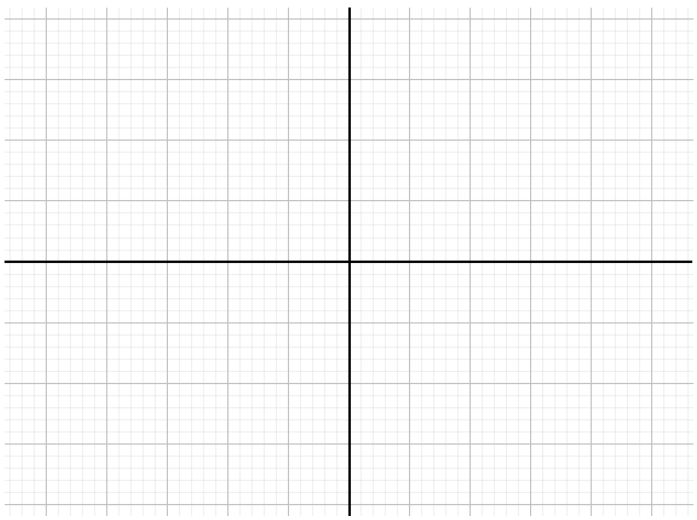
All this boils down to the *concavity test:*

- If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

Sometimes our functions will switch from concave upward to downward or the other way around. The points where this switch occurs is called an *inflection point*.

Let's look at an example.

Example 10.4 Try and sketch the graph of $f(x) = x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 6$.

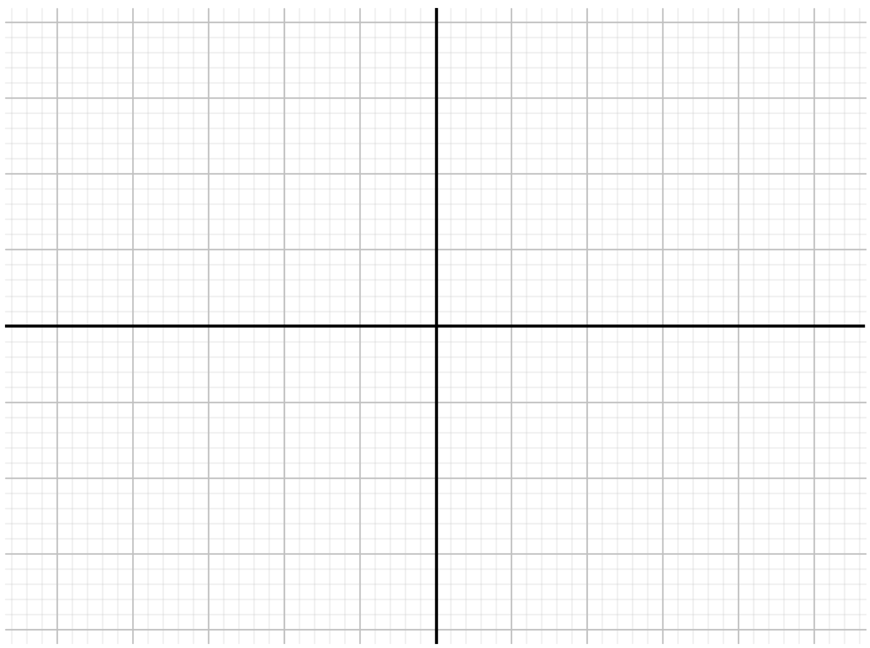


Looking at the second derivative, we can state our derivative tests in a much more concise way.

Theorem 10.5 (Second derivative test) *Suppose f'' is continuous near c.*

- *If $f'(c) = 0$ and $f''(c) > 0$, then f ahas a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f ahas a local maximum at c.*

Exercise 10.6 With a partner, sketch the graph of $f(x) =$ ________________.



10.2 L'Hôpital's Rule

We're going to go back to limits for a minute and look at how we can find limits at places that aren't that well defined. For example, what's the following limit:

$$\lim_{x\to 0} \frac{\sin(x)}{\ln(x)} = \underline{\quad}$$

Or how about

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \underline{\quad}$$

Whenever we have a limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \underline{\quad}$$

then this limit is called an ____
.

These limits aren't very easy to solve by the methods we have so far, but in 1694 the Swiss mathematician Johann Bernoulli introduced a theorem to the French mathematician Guillaume de l'Hôpital which we now call L'hôpital's rule:

Theorem 10.7 (L'Hôpital's rule) *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose further that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

(in other words, the limit is an indeterminate form of either type). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \underline{\hspace{4cm}}$$

if the limit on the right side exists or is $\pm\infty$.

This is awesome! Let's try it with our examples from before.

Example 10.8 Let $f(x) = \sin(x)$ and $g(x) = \ln(x)$ and, from above, recall

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \underline{\hspace{1cm}}$$

so we can use L'hôpitals rule!

Example 10.9 Let $f(x) = x^2$ and $g(x) = e^x$ and, from above, recall that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

We can therefore use L'Hôpital's rule!

Try another example with a partner.

Exercise 10.10 Let $f(x) = x^2 - x - 2$ and $g(x) = \sqrt{2} - \sqrt{x}$. With a partner, find

$$\lim_{x\to 2} \frac{f(x)}{g(x)}$$

Example 10.11 Which of the following can I use L'hôpital's rule on?

$$\lim_{x\to\infty} \frac{1}{x} \qquad \lim_{x\to\infty} \frac{x^2+7x}{x^2+9x+3} \qquad \lim_{x\to\infty} \frac{x^2+3x+1}{(x^4+7x)\cdot e^x}$$

$$\lim_{x\to 0} \frac{e^x}{x} \qquad \lim_{x\to\pi} \frac{\pi - x}{\cos(x)} \qquad \lim_{x\to 0} \frac{0}{\cos(x)}$$

We can also use L'Hôpital's rule in other cases! In most of these cases, what we want to do is convert our limit into a limit of one of our two indeterminate types. These are best done with examples.

The form $\infty - \infty$

Example 10.12 Find the limit

$$\lim_{x\to 0} \frac{1}{x\sin(x)} - \frac{1}{x^2}$$

$$\lim_{x\to 0} \tfrac{1}{x\sin(x)} - \tfrac{1}{x^2} \quad =$$

$$= \tfrac{1}{6}$$

Notice how we changed our formula from $\infty - \infty$ to $\frac{0}{0}$ in order to be able to use L'Hôpital's rule. We do the same for all the types.

The form $0 \cdot \pm\infty$

Example 10.13 Find the limit

$$\lim_{x\to 0} 8x^4 \ln(3x)$$

$$\lim_{x\to 0} 8x^4 \ln(3x) \quad =$$

$$= 0$$

We can do the same with powers!

The forms 0^0, ∞^0, and $1^{\pm\infty}$

In these cases, it's a little more complicated.

Example 10.14 Find the limit

$$\lim_{x \to 0} (x^2)^{x^2}$$

$$\lim_{x\to 0} (x^2)^{x^2} \quad \to$$

$$\lim_{x\to 0} (x^2)^{x^2} \quad = e^0 = 1$$

In essence, we followed the following steps:

(1) ______________________________

(2) ______________________________

(3) ______________________________

Exercise 10.15 With a partner, find the following limit.

$$\lim_{x\to 0} \sin(x)^x$$

10.3 Optimization problems - §4.7

Optimization problems are difficult. Not only because they use derivatives, but because they are generally word problems that are difficult to interpret. They are best done *slowly* and with a lot of thought to make sure that you are doing things correctly.

There are 6 basic steps to handling an optimization problem.

(1) ______________________________

(2) ______________________

(3) ________________________

(4) ____

(5) ____

(6) ______________________________

This is definitely best done with an example.

Example 10.16 Suppose we time travelled to the 8th century and got stuck in the countryside of some kingdom. Looking at our time machine we realize it's going to take up a few years until we can fix it, so we gotta make due and set-up a little farm to sustain ourselves for a little bit. After constructing a cabin that's not too shabby you realize you should probably turn the extra material into a fence to protect your crops from animals. You have material to make a 400 meter fence (you're known for super speed when it comes to chopping wood and making fences) and you can't get more material because. . . life. To help make your plot bigger you decide that one side of the rectangular plot of land will be shielded by a river that looks larger than lake ontario. What are the dimensions of your farm which give you the largest area to farm in?

Exercise 10.17 You and a partner are in charge of Coca Cola Canada (C^3) and your bosses have said they are spending to much money on metal and want to come up with a smaller size. Keeping the can a cylindrical shape, what are the dimensions of the can that minimize surface area if you have 250ml of coke in each bottle.

Some help:

- Surface area can be calculated by calculating the area of the two circles (on top and on the bottom) and then adding the surface area of the cylinder part.
- The volume of the can is the same as the area of the top times the height.
- The volume of the can is 250.