Homework 8

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Exercise 1 Find the differential of each function.

(1) $y = \frac{1+2u}{1+3u}$

(2) $y = \theta^2 \sin(2\theta)$

(3) $y = \ln(\sin(\theta))$

(4) $y = \frac{e^x}{1-e^x}$

Solution. (1) $dy = \frac{-1}{(1+3u)^2} du$

(2) $dy = 2\theta \left(\theta \cos(2\theta) + \sin(2\theta)\right) d\theta.$

(3) $dy = \cot(\theta) d\theta$

(4) $dy = \frac{e^x}{(1-e^a)^2} dx$

□

Exercise 2 Sketch the graph of f by hand and use your sketch to find the absolute/local max and min values of f.

(1) $f(x) = 2 - \frac{x}{3}$ when $x \geq -2$

(2) $f(x) = \frac{1}{x}$ when $1 < x < 3$

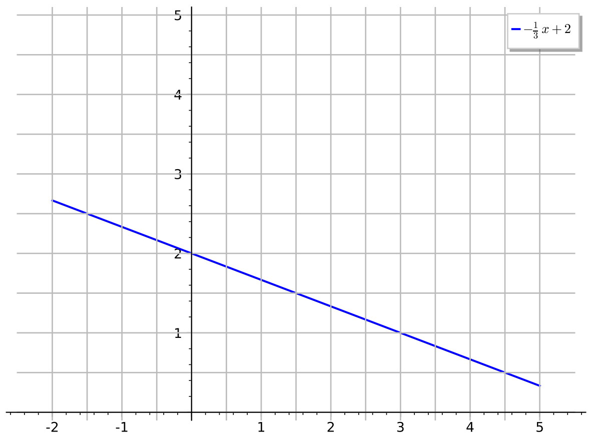
(3) $f(x) = \sin(x)$ when $0, < x \leq \pi/2$

(4) $f(x) = \cos(t)$ when $-3\pi/2 \leq t \leq 3\pi/2$

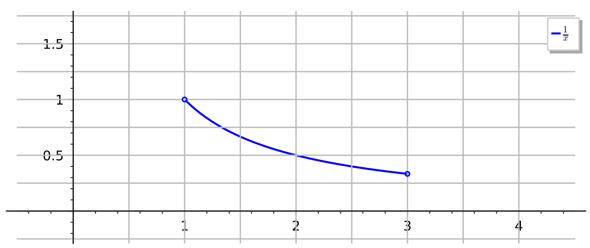
(5) $f(x) = |x|$

(6) $f(x) = e^x$

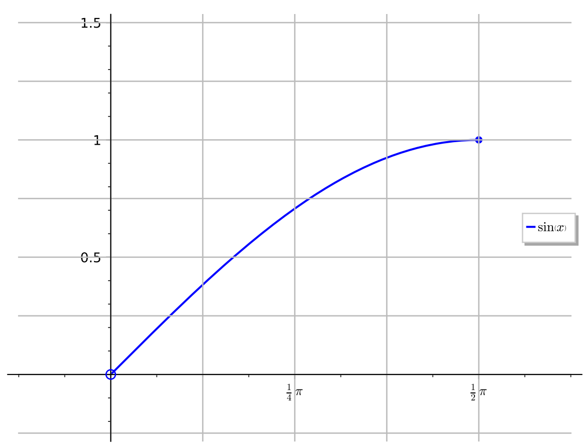
Solution. (1) Abs max at -2.



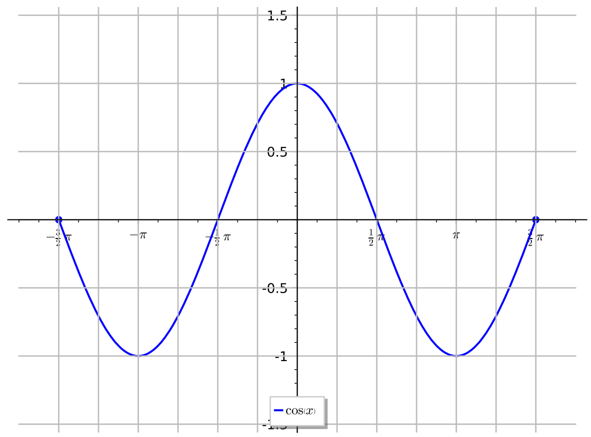
(2) No abs/local max/min.



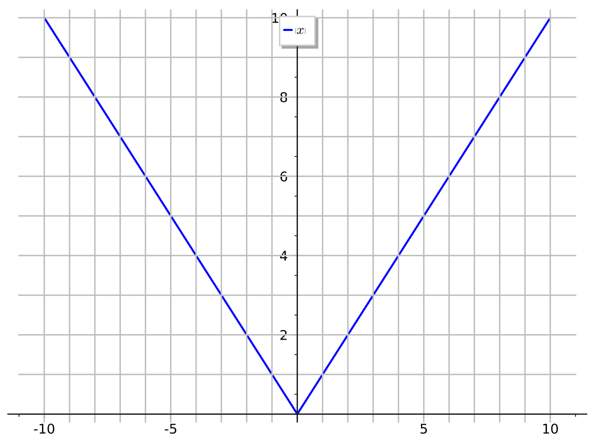
(3) Abs max at 1.



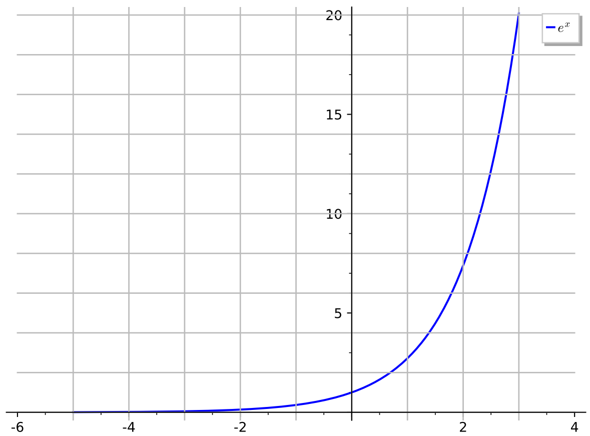
(4) Abs/local max at $f(0) = 1$ and abs/local mins at $f(\pm\pi, -1)$.



(5) No abs/loc max. Abs/loc min at $f(0) = 0$



(6) No abs/loc max nor min



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Exercise 3 Find the critical numbers of the following functions.

(1) $f(x) = x^3 + 6x^2 - 15x$

(2) $f(x) = 2x^3 + x^2 + 2x$

(3) $g(t) = |3t - 4|$

(4) $h(p) = \frac{p-1}{p^2+4}$

(5) $g(x) = \sqrt[3]{4 - x^2}$

(6) $g(\theta) = 4\theta - \tan(\theta)$

(7) $h(t) = 3t - \arcsin(t)$

(8) $f(x) = x^{-2}\ln(x)$

Solution. (1) $x = -5, 1$

(2) None

(3) $t = \frac{4}{3}$

(4) $p = 1 \pm \sqrt{5}$

(5) $x = -2, 0, 2$

(6) $\theta = \frac{\pi}{3} + 2n\pi,\ \frac{5\pi}{3} + 2n\pi,\ \frac{4\pi}{3} + 2n\pi$

(7) $t = \pm\frac{2}{3}\sqrt{2}$

(8) $x = \sqrt{e}$

□

Exercise 4 Find the abs. max and abs. min values of the following f on the given intervals.

(1) $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

(2) $f(x) = x^3 - 6x^2 + 5$, $[-3, 5]$

(3) $f(t) = (t^2 - 4)^3$, $[-2, 3]$

(4) $f(t) = \frac{\sqrt{t}}{1+t^2}$, $[0, 2]$

(5) $f(t) = t + \cot(t/2)$, $[\pi/4, 7\pi/4]$

(6) $f(x) = xe^{x/2}$, $[-3, 1]$

(7) $f(x) = x - 2\tan^{-1}(x)$, $[0, 4]$

Solution. (1) Max at $f(3) = 13$, Min at $f(0) = 5$

(2) Max at $f(0) = 5$, Min at $f(-3) = -76$

(3) Max at $f(3) = 125$, Min at $f(0) = -64$

(4) Max at $f(1) = 1$, Min at $f(0) = 0$

(5) Max at $f(\frac{1}{\sqrt{3}}) = \frac{3^{3/4}}{4}$, Min at $f(0) = 0$

(6) Max at $f(\frac{3\pi}{2}) = \frac{3\pi}{2} - 1$, Min at $f(\frac{pi}{2}) = \frac{pi}{2} + 1$

(7) Max at $f(1) = e^{1/2}$, Min at $f(-2) = \frac{-2}{e}$

(8) Max at $f(4) = 4 - 2\tan^{-1}(4)$, Min at $f(1) = 1 - \frac{pi}{2}$

□

Exercise 5 Show that $x^3 + e^x = 0$ has exactly one real root.
Solution. First, $f(-1) = -1 + 1/e < 0$ and $f(0) = 1 > 0$. Since f is the sum of a polynomial and the natural exponentiation function, f is continuous and differentiable for all x. By the intermediate value theorem, there is a number c in $(-1, 0)$ such that $f(c) = 0$. Therefore, there exists at least one real root. Now suppose there is another root a. Since f is continuous everywhere, in particular on $[a, c]$ (or $[c, a]$) then it is differentiable on (a, c) (or (c, a)). By Rolle's theorem, there is a number r in (a, c) such that $f'(r) = 0$. But $f'(r) = 3r^2 + e^r > 0$, which is a contradiction. Therefore, there is no other root.

□

Exercise 6 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the MVT.

(1) $f(x) = x^3 - 3x + 2$, $[-2, 2]$

(2) $f(x) = 1/x$, $[1, 3]$

Solution. (1) The function is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable everywhere.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Implies that $c = \frac{\pm 2}{\sqrt{3}}$ both of which are in $(-2, 2)$

(2) The function is continuous and differentiable everywhere except when $x = 0$ since we can't have 0 in the denominator. Since we are looking at the interval $[1, 3]$, it is continuous on $[1, 3]$ (which does not contain 0) and differentiable on $(1, 3)$. Therefore, using MVT we find $c = \pm\sqrt{3}$, but only $\sqrt{3}$ is between 1 and 3, therefore only $\sqrt{3}$ satisfies the conclusion of the MVT.

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