Homework 6

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An exercise marked with the symbol ⋆ is considered more difficult and will not be an exam question.

Exercise 1 Differentiate the following functions.

(1) $f(x) = x\cos(x) + 2\tan(x)$

(2) $y = 2\sec(x) - \csc(x)$

(3) $g(\theta) = e^{\theta}(\tan(\theta) - \theta)$

(4) $f(t) = \frac{\cos(t)}{e^t}$

(5) $y = \sin(\theta)\cos(\theta)$

(6) $y = \frac{\cos(x)}{1-\sin(x)}$

(7) $y = \frac{\sin(t)}{1+\tan(t)}$

(8) $f(t) = te^t\cot(t)$

Solution. (1) $f'(x) = \cos(x) - x\sin(x) + 2\sec^2(x)$

(2) $y' = 2\sec(x)\tan(x) + \csc(x)\cot(x)$

(3) $g'(\theta) = e^{\theta}\left(\sec^2(\theta) - 1 + \tan(\theta) - \theta\right)$

(4) $f'(t) = -\frac{\csc^2(t)+\cot(t)}{e^t}$

(5) $y' = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$

(6) $y' = \frac{1}{1-\sin(x)}$

(7) $y' = \frac{\cos(t)+\sin(t)-\tan(t)\sec(t)}{(1+\tan(t))^2}$

(8) $f'(t) = e^t\left(\cot(t) + t\cot(t) - t\csc^2(t)\right)$

□

Exercise 2 Prove that $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$.

Solution.

$$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{\cos(x)\cdot 0 - 1\cdot(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

□

Exercise 3 Find the following limits.

(1) $\lim_{x\to 0} \frac{\sin(x)}{\sin(\pi x)}$

(2) $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\sin(\theta)}$

(3) $\lim_{x\to 0} \frac{\sin(3x)\sin(5x)}{x^2}$

(4) ⋆ $\lim_{x\to 0} \csc(x)\sin(\sin(x))$

(5) $\lim_{x\to 0} \frac{\sin(x^2)}{x}$

(6) $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}$

Solution. (1) $\frac{1}{\pi}$

(2) 0

(3) 15

(4) 1

(5) 0

(6) $\frac{1}{3}$

□

Exercise 4 Find the derivative of the following functions.

(1) $F(x) = (1 + x + x^2)^{99}$

(2) $f(x) = \frac{1}{\sqrt[3]{x^2-1}}$

(3) $g(\theta) = \cos^2(\theta)$

(4) $f(t) = t\sin(\pi t)$

(5) $g(x) = e^{x^2-x}$

(6) $g(x) = \left(x^2 + 1\right)^3 \left(x^2 + 2\right)^6$

(7) $F(t) = (3t - 1)^4 (2t + 1)^{-3}$

(8) $y = \left(x + \frac{1}{x}\right)^5$

(9) $f(t) = 2^{t^3}$

(10) $s(t) = \sqrt{\frac{1+\sin(t)}{1+\cos(t)}}$

(11) $f(z) = e^{z/(z-1)}$

(12) $J(\theta) = \tan^2(n\theta)$

(13) $F(t) = \frac{t^2}{\sqrt{t^3+1}}$

(14) $U(y) = \left(\frac{y^4+1}{y^2+1}\right)^5$

(15) $y = x^2 e^{-1/x}$

(16) $y = \sqrt{1 + xe^{-2x}}$

(17) $y = e^{\sin(2x)} + \sin(e^{2x})$

(18) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

(19) $y = 2^{3^{4^x}}$

(20) $y = \left(x + (x + \sin^2(x))^3\right)^4$

Solution. (1) $F'(x) = 99\left(1 + x + x^2\right)^{98}(1 + 2x)$

(2) $f'(x) = \frac{-2x}{3(x^2-1)^{4/3}}$

(3) $g'(\theta) = -\sin(2\theta)$

(4) $f'(t) = \pi t \cos(\pi t) + \sin(\pi t)$

(5) $g'(x) = e^{x^2-x}(2x - 1)$

(6) $g'(x) = 6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4)$

(7) $F'(t) = 6(3t - 1)^3(2t + 1)^{-4}(t + 3)$

(8) $y' = 5\left(x + \frac{1}{x}\right)^4\left(1 - \frac{1}{x^2}\right) = \frac{5(x^2+1)^4(x^2-1)}{x^6}$

(9) $f'(t) = 3(\ln(2))t^2 2^{t^3}$

(10) $s'(t) = \frac{\cos(t)+\sin(t)+1}{2\sqrt{1+\sin(t)}(1+\cos(t))^{3/2}}$

(11) $f'(z) = -\frac{e^{z/(z-1)}}{(z-1)^2}$

(12) $J'(\theta) = 2n\tan(n\theta)\sec^2(n\theta)$

(13) $F'(t) = \frac{t(t^3+4)}{2(t^3+1)^{3/2}}$

(14) $U'(y) = \frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$

(15) $y' = e^{-1/x}(1 + 2x)$

(16) $y' = \frac{e^{-2x}(1-2x)}{2\sqrt{1+xe^{-2x}}}$

(17) $y' = 2\cos(2x)e^{\sin(2x)} + 2e^{2x}\cos(e^{2x})$

(18) $y' = \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right)$

(19) $y' = \ln(2) \cdot \ln(3) \cdot \ln(4) \cdot 4^x \cdot 3^{4^x} \cdot 2^{3^{4^x}}$

(20) $y' = 4\left(x + (x + \sin^2(x))^3\right)^3 \cdot \left(1 + 3\left(x + \sin^2(x)\right)^2 \cdot (1 + 2\sin(x)\cos(x))\right)$

□

Exercise 5 Find the first and second derivatives:

(1) $y = \frac{1}{(1+\tan(x))^2}$

(2) $y = e^{e^x}$

Solution. (1) (a) $y' = \frac{-2\sec^2(x)}{(1+\tan(x))^3}$

(b) $y'' = \frac{2\sec^2(x)\left(\tan^2(x) - 2\tan(x) + 3\right)}{(1+\tan(x))^4}$

(2) (a) $y' = e^{e^x} \cdot e^x$

(b) $y'' = e^{e^x} \cdot e^x(1 + e^x) = e^{e^x + x}(1 + e^x)$

□

Exercise 6 Find $\frac{dy}{dx}$ for the following by implicit differentiation:

(1) $2x^2 + xy - y^2 = 2$

(2) $x^3 - xy^2 + y^3 = 1$

(3) $xe^y = x - y$

(4) $\cos(xy) = 1 + \sin(y)$

(5) $e^y \sin(x) = x + xy$

(6) $xy = \sqrt{x^2 + y^2}$

(7) $x\sin(y) + y\sin(x) = 1$

(8) $\tan(x - y) = \frac{y}{1+x^2}$

Solution. (1) $y' = \frac{-4x-y}{x-2y}$

(2) $y' = \frac{y^2-3x^2}{y(3y-2x)}$

(3) $y' = \frac{1-e^y}{xe^y+1}$

(4) $y' = -\frac{y\sin(xy)}{x\sin(xy)+\cos(y)}$

(5) $y' = \frac{1+y-e^y\cos(x)}{e^y\sin(x)-x}$

(6) $y' = \frac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$

(7) $y' = \frac{-\sin(y)-y\cos(x)}{x\cos(y)+\sin(x)}$

(8) $y' = \frac{(1+x^2)\sec^2(x-y)+2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

□

Exercise 7 Find the following derivatives.

(1) $y = \arctan(x^2)$

(2) $g(x) = \arccos(\sqrt{x})$

(3) $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

(4) $R(t) = \arcsin(\frac{1}{t})$

(5) $y = \cos^{-1}\left(\sin^{-1}(t)\right)$

(6) $y = \arctan\left(\sqrt{\frac{1-x}{1+x}}\right)$

Solution. (1) $y' = \frac{2x}{1+x^4}$

(2) $g'(x) = -\frac{1}{2\sqrt{x}\sqrt{1-x}}$

(3) $y' = \frac{1}{2(1+x^2)}$

(4) $R'(t) = -\frac{1}{|t|\sqrt{t^2-1}}$

(5) $y' = -\frac{1}{\sqrt{1-(\sin^{-1}(t))^2}} \cdot \frac{1}{\sqrt{1-t^2}}$

(6) $y' = \frac{-1}{2\sqrt{1-x^2}}$

□

Exercise 8 Differentiate the following functions.

(1) $f(x) = x\ln(x) - x$

(2) $f(x) = \ln(\sin^2(x))$

(3) $y = \frac{1}{\ln(x)}$

(4) $f(x) = \log_{10}(\sqrt{x})$

(5) $g(t) = \sqrt{1+\ln(t)}$

(6) $h(x) = \ln(x + \sqrt{x^2-1})$

(7) $P(v) = \frac{\ln(v)}{1-v}$

(8) $y = \ln(|1+t-t^3|)$

(9) $y = \ln(\csc(x) - \cot(x))$

(10) $H(z) = \ln\left(\sqrt{\frac{a^2-z^2}{a^2+z^2}}\right)$

(11) $y = \log_2(x \log_5(x))$

Solution. (1) $f'(x) = \ln(x)$

(2) $f'(x) = 2\cot(x)$

(3) $y' = \frac{-1}{x(\ln(x))^2}$

(4) $f'(x) = \frac{1}{2 \cdot \ln(10) \cdot x}$

(5) $g'(t) = \frac{1}{2t\sqrt{1+\ln(t)}}$

(6) $h'(x) = \frac{1}{\sqrt{x^2-1}}$

(7) $P'(v) = \frac{1-v+v\ln(v)}{v(1-v)^2}$

(8) $y' = \frac{1-3t^2}{1+t-t^3}$

(9) $y' = \csc(x)$

(10) $H'(z) = \frac{2a^2z}{z^4-a^4}$

(11) $y' = \frac{1+\ln(x)}{x\ln(x)\ln(2)}$

□

Exercise 9 Find y' and y'' of the follwoing functions.

(1) $y = \frac{\ln(x)}{1+\ln(x)}$

(2) $y = \ln(1+\ln(x))$

Solution. (1) (a) $y' = \frac{1}{x(1+\ln(x))^2}$

(b) $y'' = -\frac{3+\ln(x)}{x^2(1+\ln(x))^3}$

(2) (a) $y' = \frac{1}{x(1+\ln(x))}$

(b) $y'' = -\frac{2+\ln(x)}{x^2(1+\ln(x))^2}$

□

Exercise 10 Differentiate the following using the logarithm method we learned in class.

(1) $y = \frac{e^{-x}\cos^2(x)}{x^2+x+1}$

(2) $y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}$

(3) $y = x^{\cos(x)}$

(4) $y = \sqrt{x}^x$

(5) $y = (\sin(x))^{\ln(x)}$

(6) $y = (\ln(x))^{\cos(x)}$

Solution. (1) $y' = -\frac{e^{-x}\cos^2(x)}{x^2+x+1}\left(1 + 2\tan(x) + \frac{2x+1}{x^2+x+1}\right)$

(2) $y' = \sqrt{x}e^{x^2-x}(x+1)^{2/3}\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3x+3}\right)$

(3) $y' = x^{\cos(x)}\left(\frac{\cos(x)}{x} - \ln(x)\sin(x)\right)$

(4) $y' = \sqrt{x}^x\,(1 + \ln(x))$

(5) $y' = (\sin(x))^{\ln(x)}\left(\ln(x)\cot(x) + \frac{\ln(\sin(x))}{x}\right)$

(6) $y' = (\ln(x))^{\cos(x)}\left(\frac{\cos(x)}{x\ln(x)} - \sin(x)\ln(\ln(x))\right)$

□

Exercise 11 ($\star$) Find y' if $y^x = x^y$.

Solution.

$$y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}$$

□

Exercise 12 Find the 9th derivative of $f(x) = x^8\ln(x)$.

Solution.

$$\frac{d^9}{dx^9}f(x) = f^{(9)}(x) = \frac{8!}{x}$$

□