Homework 3 solutions

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Exercise 1 Prove the following limit using an ε, δ proof.

$$\lim_{x\to 0} x^3 = 0$$

Solution. (1) **Analysis:** Let's start with

$$|x^3 - 0| < \varepsilon$$

and try and change the equation to something like $|x - 0| < \delta$. Then

$$\begin{aligned} |x^3 - 0| &< \varepsilon \\ |x^3| &< \varepsilon \\ |x|^3 &< \varepsilon \\ |x| &< \sqrt[3]{\varepsilon} \end{aligned}$$

Therefore we want to let $\delta = \sqrt[3]{\varepsilon}$.

(2) **Proof:** Suppose $\varepsilon > 0$ and let $\delta = \sqrt[3]{\varepsilon} > 0$. If $0 < |x - 0| < \delta$ then

$$|x^3 - 0| = |x|^3 < \delta^3 = \sqrt[3]{\varepsilon}^3 = \varepsilon.$$

(3) Therefore, by the definition of a limit we are done.

□

Exercise 2 Prove the following limit using an ε, δ proof.

$$\lim_{x\to -6^+} \sqrt[8]{6+x} = 0$$

Solution. (1) **Analysis:** Start with

$$|\sqrt[8]{6+x} - 0| < \varepsilon$$

and we want to change this equation to look something like $-6 < x < -6 + \delta$.

$$\begin{aligned} |\sqrt[8]{6+x} - 0| &< \varepsilon \\ |\sqrt[8]{6+x}| &< \varepsilon \\ 0 < \sqrt[8]{6+x} &< \varepsilon \\ 0^8 < 6 + x &< \varepsilon^8 \\ 0 - 6 < x &< \varepsilon^8 - 6 \end{aligned}$$

Therefore, we want to let $\delta = \varepsilon^8$.

(2) **Proof:** Suppose $\varepsilon > 0$ and let $\delta = \varepsilon^8 > 0$. If $-6 < x < \delta - 6$ then

$$\begin{aligned} & -6 < x < \delta - 6 \\ \Rightarrow & 0 < x + 6 < \delta = \varepsilon^8 \\ \Rightarrow & 0 < \sqrt[8]{x+6} < \varepsilon \\ \Rightarrow & |\sqrt[8]{x+6}| < \varepsilon \end{aligned}$$

(3) Therefore, by definition of the right-hand limit we are done.

□

Exercise 3 Evaluate the following limits and justify each step.

(1)

$$\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$$

(2)

$$\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$$

(3)

$$\lim_{t \to 2} \left(\frac{t^2 - 2}{t^3 - 3t + 5} \right)^2$$

Solution. (1)

$$\begin{aligned} \lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3) &= \left(\lim_{x \to -1} x^4 - 3x \right) \left(\lim_{x \to -1} x^2 + 5x + 3 \right) \\ &= \left(\lim_{x \to -1} x^4 - \lim_{x \to -1} 3x \right) \left(\lim_{x \to -1} x^2 + \lim_{x \to -1} 5x + \lim_{x \to -1} 3 \right) \\ &= \left((-1)^4 - 3(-1) \right) \left((-1)^2 + 5(-1) + 3 \right) \\ &= (1 + 3)(1 - 5 + 3) \\ &= 4 \cdot -1 \\ &= -4 \end{aligned}$$

(2)

$$\begin{aligned} \lim_{u \to -2} \sqrt{u^4 + 3u + 6} &= \sqrt{\lim_{u \to -2} u^4 + 3u + 6} \\ &= \sqrt{\lim_{u \to -2} u^4 + \lim_{u \to -2} 3u + \lim_{u \to -2} 6} \\ &= \sqrt{(-2)^4 + 3(-2) + 6} \\ &= \sqrt{16 - 6 + 6} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

(3)

$$\begin{aligned}\lim_{t\to 2}\left(\frac{t^2-2}{t^3-3t+5}\right)^2 &= \left(\lim_{t\to 2}\frac{t^2-2}{t^3-3t+5}\right)^2\\ &= \left(\frac{\lim_{t\to 2} t^2-2}{\lim_{t\to 2} t^3-3t+5}\right)^2\\ &= \left(\frac{2^2-2}{2^3-3(2)+5}\right)^2\\ &= \left(\frac{4-2}{8-6+5}\right)^2\\ &= \left(\frac{2}{7}\right)^2\\ &= \frac{4}{49}\end{aligned}$$

□

Exercise 4 Evaluate the following limits, if they exist.

(1)

$$\lim_{x\to -3}\frac{x^2+3x}{x^2-x-12}$$

(2)

$$\lim_{x\to 4}\frac{x^2+3x}{x^2-x-12}$$

(3)

$$\lim_{x\to -1}\frac{2x^2+3x+1}{x^2-2x-3}$$

(4)

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h}$$

(5)

$$\lim_{t\to 1}\frac{t^4-1}{t^3-1}$$

(6)

$$\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}$$

(7)

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}$$

(8)

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$$

(9)

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^2-3x^2-4}$$

(10)

$$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}$$

(11)

$$\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-1\frac{1}{x^2}}{h}$$

Solution. (1) $\frac{3}{7}$

(2) DNE

(3) $\frac{1}{4}$

(4) 12

(5) $\frac{4}{3}$

(6) $\frac{2}{3}$

(7) $-\frac{1}{9}$

(8) 1

(9) 0

(10) $-\frac{4}{5}$

(11) $-\frac{2}{x^3}$

□

Exercise 5 Use the squeeze theorem to show

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin(\frac{\pi}{x})=0$$

Solution. First recall that $-1\leq\sin(\frac{\pi}{x})\leq 1$. Therefore

$$-\sqrt{x^3+x^2}\leq\sqrt{x^3+x^2}\sin(\frac{\pi}{x})\leq\sqrt{x^3+x^2}$$

But, by the limit laws we have:

$$\lim_{x\to 0}-\sqrt{x^3+x^2}=-\sqrt{(-1)^3+(-1)^2}=-\sqrt{-1+1}=0$$

and

$$\lim_{x\to 0} \sqrt{x^3+x^2} = \sqrt{(-1)^3+(-1)^2} = \sqrt{-1+1} = 0$$

Therefore, by the squeeze theorem

$$\lim_{x\to 0} \sqrt{x^3+x^2}\sin(\frac{\pi}{x}) = 0$$

□

Exercise 6 If $2x \le g(x) \le x^4 - x^2 + 2$, for all x, find $\lim_{x\to 1} g(x)$.
Solution. Since $\lim_{x\to 1} 2x = 2$ and $\lim_{x\to 1} x^4 - x^2 + 2 = 1 - 1 + 2 = 2$, therefore, by the squeeze theorem, $\lim_{x\to 1} g(x) = 2$. □

Exercise 7 Find the following limits, if they exist.

(1)
$$\lim_{x\to -6} \frac{2x+12}{|x+6|}$$

(2)
$$\lim_{x\to -2} \frac{2-|x|}{2+x}$$

(3)
$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$$

Solution. (1) DNE

(2) 1

(3) 0

□

Exercise 8 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(1) $g(t) = \frac{t^2+5t}{2t+1}$, $a = 2$

(2) $f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}$, $a = 2$

Solution. (1) It suffices to show $\lim_{t\to 2} g(t) = \frac{14}{5} = g(2)$.

(2) It suffices to show $\lim_{x\to 2} f(x) = 40 = f(2)$.

□

Exercise 9 Explain why the functions below iare discontinous at the given number a.

(1) for $a = -2$,

$$f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x \neq -2 \\ 2^x & \text{if } x = 2 \end{cases}$$

(2) for $a = -1$,

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

(3) for $a = 3$,

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Solution. (1) The limit $\lim_{x\to -2} f(x)$ does not exist.

(2) We don't have equality:

$$\lim_{x\to 1} f(x) = \frac{1}{2} \neq 1 = f(1)$$

(3) We don't have equality:

$$\lim_{x\to 3} f(x) = 7 \neq 6 = f(3)$$

□

Exercise 10 Explain using our theorems, why the functions below are continuous everywhere on its domain. What are their domains?

(1) $G(x) = \frac{x^2+1}{2x^2-x-1}$

(2) $R(t) = \frac{e^{\sin(t)}}{2+\cos(\pi t)}$

(3) $B(x) = \frac{\tan(x)}{\sqrt{4-x^2}}$

(4) $N(r) = \tan^{-1}(1 + e^{-r^2})$

Solution. (1) $G(x)$ is a rational function so it is continuous on its domain: $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$.

(2) Since $e^{\sin(t)}$ and $\cos(\pi t)$ are continuous everywhere then $2 + \cos(\pi t)$ is continuous everywhere and so is $R(t) = \frac{e^{\sin(t)}}{2+\cos(\pi t)}$. Therefore the domain is $\mathbb{R}$.

(3) We know that $\tan(x)$ is continuous on its domain: $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. Furthermore, $\sqrt{4-x^2}$ is continuous on its domain: $[-2, 2]$. Finally, we have $B(x)$ is continuous on its domain: $(-2, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}), \cup(\frac{\pi}{2}, 2)$.

(4) By our theorems e^{-r^2} is continuous everywhere and therefore, the composite function $1 + e^{-r^2}$ is continuous everywhere. Furthermore, $\tan^{-1}$ is continuous everywhere and therefore $N(r)$ is continuous everywhere on $\mathbb{R}$.

□

Exercise 11 Locate the dicontinuities of $y = \ln(\tan^2(x))$.

Solution. First, $\tan^2(x)$ is discontinuous at $x = \frac{\pi}{2} + \pi k$ where k is any integer. Furthermore, ln is discontinuous whenever $\tan^2(x) = 0$, in other words whenever $x = \pi k$. Therefore, our function is discontinous at $x = \frac{\pi}{2}n$ for any integer n. □

Exercise 12 Use the intermediate value theorem to show that there is a root of the given equation in the specified interval.

(1) $\ln(x) = x - \sqrt{x}$ on $(2, 3)$.

(2) $\sin(x) = x^2 - x$ on $(1, 2)$.

Solution. (1) Since $\ln(x) = x - \sqrt{x}$ is equivalent to $\ln(x) - x + \sqrt{x} = 0$, we let $f(x) = \ln(x) - x + \sqrt{x}$. Since $f(x)$ is continuous on $[2, 3]$ we can plug in the end points of our interval:

$$f(2) = \ln(2) - 2 + \sqrt{2} \approx 0.107 \quad f(3) = \ln(3) - 3 + \sqrt{3} \approx -0.169$$

Therefore, since $f(2) > 0 > f(3)$, by the intermediate value theorem, there is a number c in $(2, 3)$ such that $f(c) = 0$. Thus, there is a root in the interval $(2, 3)$ for the equation $\ln(x) = x - \sqrt{x}$.

(2) Since $\sin(x) = x^2 - x$ is equivalent to $\sin(x) - x^2 + x = 0$, we let $f(x) = \sin(x) - x^2 + x$. Since $f(x)$ is continuous on the interval $[1, 2]$, we can plug in the end points of our interval:

$$f(1) = \sin(1) - 1^2 + 1 \approx 0.84 \quad f(2) = \sin(2) - 2^2 + 2 \approx -1.09$$

Therefore, since $f(1) > 0 > f(2)$, by the intermediate value theorem, there is a number c in $(1, 2)$ such that $f(c) = 0$. Thus, there is a root in the interval $(1, 2)$ for the equation $\sin(x) = x^2 - x$.

□