Homework 2 solutions

by Aram Dermenjian

13 January 2020

An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Exercise 1 Find the exact value of each expression.

(1) $\log_5(\frac{1}{125})$

(2) $\ln(\frac{1}{e^2})$

(3) $e^{-\ln(2)}$

(4) $e^{\ln(\ln(e^3))}$

Solution. (1)

$$\log_5(\frac{1}{125}) = \log_5(\frac{1}{5^3}) = \log_5(5^{-3}) = -3\log_5(5) = -3$$

(2)

$$\ln(1/e^2) = \ln(e^{-2}) = -2\ln(e) = -2$$

(3)

$$e^{-\ln(2)} = \frac{1}{e^{\ln(2)}} = \frac{1}{2}$$

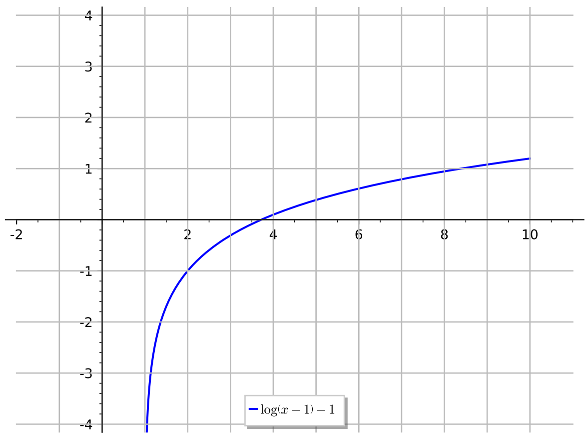
(4)

$$e^{\ln(\ln(e^3))} = e^{\ln(3\ln(e))} = e^{\ln(3)} = 3$$

□

Exercise 2 Let $f(x) = \ln(x-1) - 1$. Sketch the graph of f and find the following:

- The domain
- The range
- The x-intercept



Solution.

(1) Domain: $x > 1$.

(2) Range: $\mathbb{R}$

(3) x-intercept: We want $f(x) = 0$. Therefore:

$$\begin{aligned} \ln(x-1) - 1 &= 0 \\ \ln(x-1) &= 1 \\ e^{\ln(x-1)} &= e^1 \\ x - 1 &= e \\ x &= e + 1 \end{aligned}$$

□

Exercise 3 Solve each equation for x:

(1) $\ln(x^2 - 1) = 3$

(2) $e^{2x} - 3e^x + 2 = 0$

(3) $\ln(\ln(x)) = 1$

(4) $e^{ax} = Ce^{bx}$ where $a \neq b$.

Solution. (1)

$$\begin{aligned} \ln(x^2 - 1) &= 3 \\ x^2 - 1 &= e^3 \\ x^2 &= e^3 + 1 \\ x &= \pm\sqrt{e^3 + 1} \end{aligned}$$

(2)

$$\begin{aligned} e^{2x} - 3e^x + 2 &= 0 \\ (e^x - 1)(e^x - 2) &= 0 \\ e^x &= 1 \text{ or } e^x = 2 \\ x &= \ln(1) = 0 \text{ or } x = \ln(2) \end{aligned}$$

(3)

$$\begin{aligned} \ln(\ln(x)) &= 1 \\ \ln(x) &= e \\ x &= e^e \end{aligned}$$

(4)

$$\begin{aligned} e^{ax} &= Ce^{bx} \\ ax &= \ln\left(Ce^{bx}\right) \\ &= \ln\left(C\right) + \ln\left(e^{bx}\right) \\ &= \ln(C) + bx \\ ax - bx &= \ln(C) \\ x(a-b) &= \ln(C) \\ x &= \frac{\ln(C)}{a-b} \end{aligned}$$

□

Exercise 4 Find the exact value of each expression.

(1) $\tan^{-1}(\sqrt{3})$

(2) $\arctan(-1)$

(3) $\sin^{-1}(\frac{-1}{\sqrt{2}})$

(4) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(5) $\arcsin(\sin(5\pi/4))$

(6) $\cos(2\sin^{-1}(\frac{5}{13}))$

Solution. (1) $\frac{\pi}{3}$

(2) $-\frac{\pi}{4}$

(3) $-\frac{\pi}{4}$

(4) $\frac{\pi}{6}$

(5) $-\frac{\pi}{4}$

(6) $\frac{119}{169}$

□

Exercise 5 Explain in your own words what is meant by the equation:

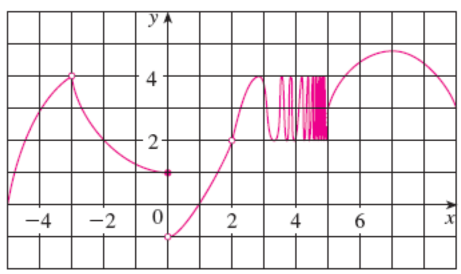
$$\lim_{x\to 2} f(x) = 5.$$

Is it possible for this statement to be true and yet $f(2) = 3$?

Solution. As x approaches 2, from either side, then $f(x)$ gets closer and closer to 5.

Yes, $f(2) = 3$ is possible. Although $f(x)$ might get closer to 5 as x approaches 2, the limit never looks at the actual value of $f(x)$ at 2. The graph could have a hole in it at $x = 2$. □

Exercise 6 Suppose the function f has the following graph:



State the value of each quantity if it exists and if it doesn't exist, explain why.

(1)
$$\lim_{x\to -3^-} f(x)$$

(2)
$$\lim_{x\to -3^+} f(x)$$

(3)
$$\lim_{x\to -3} f(x)$$

(4) $f(-3)$

(5)
$$\lim_{x\to 0^-} f(x)$$

(6)
$$\lim_{x\to 0^+} f(x)$$

(7)
$$\lim_{x \to 0} f(x)$$

(8) $f(0)$

(9)
$$\lim_{x \to 2} f(x)$$

Solution. (1) 4

(2) 4

(3) 4

(4) There is a hole at -3 and therefore $f(-3)$ is not defined.

(5) 1

(6) -1

(7) Since the right hand and left hand limits are different, the limit itself does not exist.

(8) 1

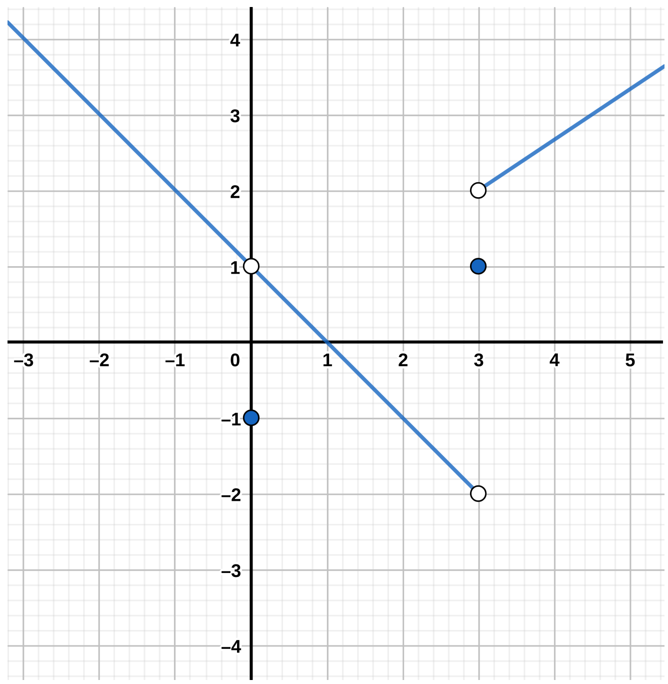
(9) There is a hole at 2 and therefore $f(2)$ does not exist.

□

Exercise 7 Sketch the graph of an example of a function f that satisfies all the given conditions.

- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 3^-} f(x) = -2$
- $\lim_{x \to 3^+} f(x) = 2$
- $f(0) = -1$
- $f(3) = 1$

Solution. Here is an example; your example may differ.



□

Exercise 8 Guess the value of the limit (if it exists) by evaluating the function at the given numbers using a calculator.

(1)

$$\lim_{x\to -3} \frac{x^2 - 3x}{x^2 - 9}$$

for

$$x = -2.5,\ -2.9,\ -2.95,\ -2.99,\ -2.999,\ -2.9999,\ -3.5,\ -3.1,\ -3.01,\ -3.001,\ -3.0001$$

(2)

$$\lim_{h\to 0} \frac{(2+h)^5 - 32}{h}$$

for

$$x = \pm 0.5,\ \pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$$

Solution. (1)

x	f(x)	x	f(x)
-2.5	-5	-3.5	7
-2.9	-29	-3.1	31
-2.95	-59	-3.-5	61
-2.99	-299	-3.01	301
-2.999	-2999	-3.001	3001
-2.9999	-29999	-3.0001	30001

So it appears:

$$\lim_{x\to 3^+} f(x) = -\infty \qquad \lim_{x\to 3^-} f(x) = \infty \qquad \lim_{x\to 3} f(x) = DNE$$

(2)

x	f(x)	x	f(x)
0.5	131.312500	-0.5	48.812500
0.1	88.410100	-0.1	72.390100
0.01	80.804010	-0.01	79.203990
0.001	80.080040	-0.001	79.920040
0.0001	80.008000	-0.0001	79.992000

So it appears that

$$\lim_{h\to 0} \frac{(2+h)^5 - 32}{h} = 80$$

□

Exercise 9 Use a table of values to estimate the value of the limit for the following limits.

(1)

$$\lim_{p\to -1} \frac{1+p^9}{1+p^15}$$

(2)

$$\lim_{t\to 0} \frac{5^t - 1}{t}$$

(3)

$$\lim_{x\to 0^+} x^2 \ln(x)$$

Solution. (1) 0.6

(2) ≈ 1.6094

(3) 0

□

Exercise 10 Determine the infinite limit in the following limits.

(1)

$$\lim_{x\to 5^-} \frac{x+1}{x-5}$$

(2)

$$\lim_{x\to 3^-} \frac{\sqrt{x}}{(x-3)^5}$$

(3)

$$\lim_{x\to 0^+} \ln(\sin(x))$$

(4)

$$\lim_{x \to \pi^-} \cot(x)$$

(5)

$$\lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$$

(6)

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \ln(x) \right)$$

Solution. (1) $-\infty$

(2) $-\infty$

(3) $-\infty$

(4) $-\infty$

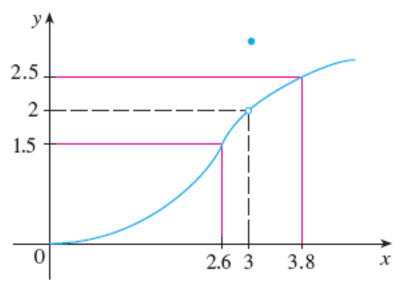
(5) $-\infty$

(6) ∞

□

Exercise 11 Use the given graph of f to find a number δ such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |f(x) - 2| < 0.5$$



Solution. We can let δ be any positive number smaller than 0.4. □

Exercise 12 Prove the following limits using an ε, δ proof.

(1)

$$\lim_{x \to 10} \left(3 - \frac{4}{5}x \right) = -5$$

(2)

$$\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$$

(3)

$$\lim_{x \to a} c = c$$

Solution. (1) (a) **Analysis:** Let's start with

$$|f(x) - L| = |3 - \frac{4}{5}x - (-5)| < \varepsilon$$

and try and change the equation to something like $|x - 10| < \delta$.

$$\begin{aligned} |3 - \frac{4}{5}x + 5| &= |8 - \frac{4}{5}x| \\ &= \frac{4}{5}|10 - x| \\ &= \frac{4}{5}|x - 10| < \varepsilon \\ &\Rightarrow |x - 10| < \frac{5}{4}\varepsilon \end{aligned}$$

Therefore, we will let $\delta = \frac{5}{4}\varepsilon$ in the proof.

(b) **Proof:** Suppose that $\varepsilon > 0$ and let $\delta = \frac{5}{4}\varepsilon > 0$. If $0 < |x - 10| < \delta$, then

$$|3 - \frac{4}{5}x + 5| = \frac{4}{5}|10 - x| < \frac{4}{5}\delta = \frac{4}{5}\frac{5}{4}\varepsilon = \varepsilon.$$

Therefore, by the definition of a limit, our limit is valid.

(2) (a) **Analysis:** Let's start with

$$|f(x) - L| = |\frac{9 - 4x^2}{3 + 2x} - 6| < \varepsilon$$

and try and change the equation to something like $|x - (-1.5)| < \delta$.

$$\begin{aligned} |\frac{9 - 4x^2}{3 + 2x} - 6| &= |\frac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6| \\ &= |3 - 2x - 6| \\ &= |-2x - 3| \\ &= 2|x + 1.5| < \varepsilon \\ &\Rightarrow |x + 1.5| < \frac{\varepsilon}{2} \end{aligned}$$

Therefore we will let $\delta = \frac{\varepsilon}{2}$ in the proof.

(b) **Proof:** Suppose that $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2} > 0$. If $0 < |x + 1.5| < \delta$, then

$$|\frac{9 - 4x^2}{3 + 2x} - 6| = 2|x + 1.5| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

Therefore, by the definition of a limit, our limit is valid.

(3) (a) **Analysis:** Let's start with

$$|f(x) - L| = |c - c| < \varepsilon$$

and try and change the equation to something like $|x - a|$. Since $|c - c| = 0$ then we can just let $\varepsilon = \delta$.

(b) **Proof:** Suppose that $\varepsilon > 0$ and let $\delta = \varepsilon$. Since $\varepsilon > 0$ then $\varepsilon > |c - c|$ for any c. But this was what we wanted to show. Therefore, by the definition of a limit, our limit is valid.

$\square$