Homework 1 solutions

by Aram Dermenjian

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

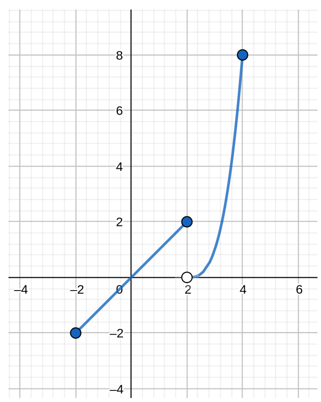
Exercise 1 Let $f(x) = x + \sqrt{4-x} + x^2$ and $g(u) = u + \sqrt{4-u} + u^2$. Is it true that $f = g$?

Solution. Yes, since both functions are exactly the same. The variable chosen to represent the function makes no difference. □

Exercise 2 Let $f(x) = \frac{x^2-x}{x-1}$ and $g(x) = x$. Is it true that $f = g$?

Solution. No, since g has as a domain $\mathbb{R}$ and f has as a domain $\mathbb{R}\setminus\{0\}$. □

Exercise 3 Let f be the function with the following graph:



What is the domain and range of f?

Solution. The domain is $[-2, 4]$ as these are the only values of x for which this function is defined. The range is $[-2, 8]$ as these are the only values of $f(x)$ which the function achieves. □

Exercise 4 Give the domain and range of each of the following functions.

(1) $f(x) = \sqrt{x}$

(2) $f(x) = \sqrt{x+1}$

(3) $f(x) = \frac{1}{x}$

(4) $f(x) = \frac{1}{1-x^2}$

Solution. (1) Domain: $[0, \infty)$ – all non-negative real numbers.
Range: $[0, \infty)$ – all non-negative real numbers.

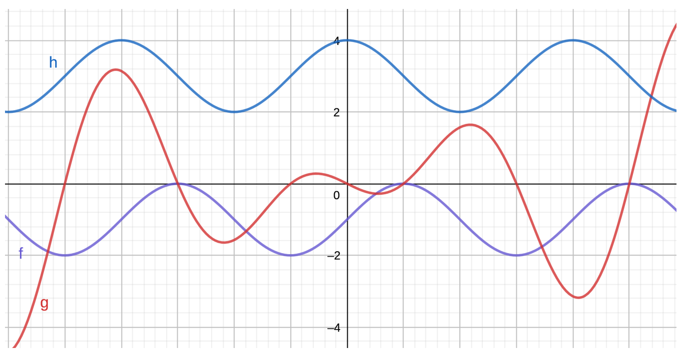
(2) Domain: $[-1, \infty)$ – For $\sqrt{x+1}$ to make sense, we must have $x + 1 \geq 0$, or $x \geq -1$, giving us the domain.
Range: $[0, \infty)$ – all non-negative real numbers.

(3) Domain: $\mathbb{R}\setminus\{0\}$ – all non-zero numbers since $\frac{1}{x}$ only doesn't make sense when $x = 0$ as we can't divide by 0.
Range: $\mathbb{R}\setminus\{0\}$ – Since any non-zero real number x can be written as $\frac{1}{\frac{1}{x}}$ therefore, every non-zero real number is in our range.

(4) Domain: $\mathbb{R}\setminus\{\pm 1\}$ – Since we can't divide by 0, we can't have $1 - x^2 = 0$, in other words x can't be -1 or 1.
Range: $(-\infty, 0) \cup [1, \infty)$ – If $x < -1$ then $f(x) < 0$. Similarly, if $x > 1$ then $f(x) < 0$. On the other hand, if $-1 < x < 1$ then $x \geq 1$.

□

Exercise 5 Let f, g and h be the functions with the following graphs:



For each, determin whether the function is even, odd or neither.
Solution. The function f is neither even nor odd. The function g is odd. The function h is even. □

Exercise 6 Determin whether the function f is even, odd or neither.

(1) $f(x) = \frac{x}{x^2+1}$

(2) $f(x) = \frac{x^2}{x^4+1}$

(3) $f(x) = 1 + 3x^3 - x^5$

(4) $f(x) = \frac{x}{x+1}$

Solution. (1) Odd

(2) Even

(3) Neither

(4) Neither

□

Exercise 7 Find the domain of the following function.

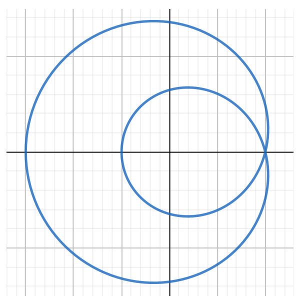
$$q(y) = \frac{|y+1|}{(y-1)^2}$$

Solution. Since $|y+1|$ is defined everywhere, it doesn't effect our domain. Since we can't divide by 0, we can't have $(y-1)^2 = 0$. In otherwords

$$\begin{aligned}(y-1)^2 &\neq 0\\ |y-1| &\neq 0\\ y &\neq 1.\end{aligned}$$

So the domain is given by $\mathbb{R}\setminus\{1\}$. □

Exercise 8 Consider the following graph:



Is this the graph of a function and why?
Solution. This is not the graph of a function because it does not pass the vertical line test. In other words, one number gets mapped to multiple points, which is not allowed. □

Exercise 9 Let $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$. Find $f+g$, $f-g$, fg, f/g and state their domains.
Solution. (1) $(f+g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$ with domain $(-\infty,-1] \cup [1,3]$.

(2) $(f-g)(x) = \sqrt{3-x} - \sqrt{x^2-1}$ with domain $(-\infty,-1] \cup [1,3]$.

(3) $(fg)(x) = \sqrt{3-x} \cdot \sqrt{x^2-1}$ with domain $(-\infty,-1] \cup [1,3]$.

(4) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{3-x}}{\sqrt{x^2-1}} = \sqrt{\frac{3-x}{x^2-1}}$ with domain $(-\infty,-1) \cup (1,3]$.

□

Exercise 10 Find the functions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$ and their domains for the following:

(1) $f(x) = x^3 - 2$, $g(x) = 1 - 4x$

(2) $f(x) = \sin(x)$, $g(x) = x^2 + 1$

(3) $f(x) = \frac{x}{x+1}$, $g(x) = \sin(2x)$

Solution. (1)
- $(f \circ g)(x) = -1 - 12x + 48x^2 - 64x^3$.
- $(g \circ f)(x) = 9 - 4x^3$.
- $(f \circ f)(x) = x^9 - 6x^6 + 12x^3 - 10$.

- $(g \circ g)(x) = -3 + 16x$.
- The domain of all of the above is $\mathbb{R}$.

(2)
- $(f \circ g)(x) = \sin\left(x^2 + 1\right)$.
- $(g \circ f)(x) = \sin^2(x) + 1$.
- $(f \circ f)(x) = \sin(\sin(x))$.
- $(g \circ g)(x) = x^4 + 2x^2 + 2$.
- THe domain of all the above is $\mathbb{R}$.

(3)
- $(f \circ g)(x) = \frac{\sin(2x)}{1+\sin(2x)}$ with domain $\mathbb{R}\setminus\left\{\frac{3\pi}{4} + \pi n \mid n \in \mathbb{Z}\right\}$.
- $(g \circ f)(x) = \sin\left(\frac{2x}{1+x}\right)$ with domain $\mathbb{R}\setminus\{-1\}$.
- $(f \circ f)(x) = \frac{x}{2x+1}$ with domain $\mathbb{R}\setminus\left\{-1, -\frac{1}{2}\right\}$.
- $(g \circ g)(x) = \sin(2 \cdot \sin(2x))$ with domain $\mathbb{R}$.

□

Exercise 11 Find $f \circ g \circ h$ where

(1) $f(x) = |x - 4|$, $g(x) = 2^x$, $h(x) = \sqrt{x}$.

(2) $f(x) = \tan(x)$, $g(x) = \frac{x}{x-1}$, $h(x) = \sqrt[3]{x}$.

Solution. (1) $(f \circ g \circ h)(x) = |2^{\sqrt{x}} - 4|$.

(2) $(f \circ g \circ h)(x) = \tan\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right)$.

□

Exercise 12 ($\star$) Suppose g is an even function and let $h = f \circ g$. Is h always an even function.

Solution. Yes! Since:

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x) = h(x)$$

□

Exercise 13 ($\star$) Suppsoe g is an odd function and let $h = f \circ g$. Is h always an odd function?

Solution. No. Since if f is odd then

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x))$$

implying h is odd. If f is even then

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x))$$

implying h is even. □

Exercise 14 Let $f(x) = 2x^2 - 1$ and $g(x) = 4x^3 - 3x$. Show that $f \circ g = g \circ f$.

Solution. It suffices to show that

$$(f \circ g)(x) = (g \circ f)(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

□

Exercise 15 Express the following in teh form of $f \circ g$.

(1) $F(x) = \cos^2(x)$.

(2) $q(x) = \sqrt[3]{\frac{x}{1+x}}$.

(3) $u(t) = \frac{\tan(t)}{1+\tan(t)}$.

Solution. (1) $f(x) = x^2$, $g(x) = \cos(x)$.

(2) $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x}{1+x}$.

(3) $f(t) = \frac{t}{1+t}$, $g(t) = \tan(t)$.

□

Exercise 16 Find the exact trigonometric ratios for the angle whose radian measure is given:

(1) $\frac{4\pi}{3}$.

(2) -5π.

(3) $\frac{11\pi}{4}$.

i.e., find sin, cos, tan, csc, sec and cot.

Solution. (1)
- $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.
- $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$.
- $\tan\left(\frac{4\pi}{3}\right) = \sqrt{3}$.
- $\csc\left(\frac{4\pi}{3}\right) = -\frac{2}{\sqrt{3}}$.
- $\sec\left(\frac{4\pi}{3}\right) = -2$.
- $\cot\left(\frac{4\pi}{3}\right) = \frac{1}{\sqrt{3}}$.

(2)
- $\sin(-5\pi) = 0$.
- $\cos(-5\pi) = -1$.
- $\tan(-5\pi) = 0$.
- $\csc(-5\pi) =$ undefined.
- $\sec(-5\pi) = -1$.
- $\cot(-5\pi) =$ undefined.

(3)
- $\sin\left(\frac{11\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

- $\cos\left(\frac{11\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.
- $\tan\left(\frac{11\pi}{4}\right) = -1$.
- $\csc\left(\frac{11\pi}{4}\right) = \sqrt{2}$.
- $\sec\left(\frac{11\pi}{4}\right) = -\sqrt{2}$.
- $\cot\left(\frac{11\pi}{4}\right) = -1$.

□

Exercise 17 Find the remaining trigonometric ratios.

(1) $\cos(x) = -\frac{1}{3}$, where $\pi < x < \frac{3\pi}{2}$.

(2) $\csc(\theta) = -\frac{4}{3}$, where $\frac{3\pi}{2} < \theta < 2\pi$.

Solution. (1)
- $\sin(x) = -\frac{2\sqrt{2}}{3}$.
- $\tan(x) = 2\sqrt{2}$.
- $\csc(x) = -\frac{3}{2\sqrt{2}}$.
- $\sec(x) = -3$.
- $\cot(x) = \frac{1}{2\sqrt{2}}$.

(2)
- $\sin(\theta) = -\frac{3}{4}$.
- $\cos(\theta) = \frac{\sqrt{7}}{4}$.
- $\tan(\theta) = -\frac{3}{\sqrt{7}}$.
- $\sec(\theta) = \frac{4}{\sqrt{7}}$.
- $\cot(\theta) = -\frac{\sqrt{7}}{3}$.

□

Exercise 18 Show that $\sin^2(x) - \sin^2(y) = \sin(x+y)\sin(x-y)$

Solution.

$$\begin{aligned}
\sin(x+y)\sin(x-y) &= (\sin(x)\cos(y) + \cos(x)\sin(y))(\sin(x)\cos(y) - \cos(x)\sin(y)) \\
&= \sin^2(x)\cos^2(y) - \sin(x)\cos(y)\cos(x)\sin(y) + \cos(x)\sin(y)\sin(x)\cos(y) - \cos^2(x)\sin^2(y \\
&= \sin^2(x)\cos^2(y) - \cos^2(x)\sin^2(y) \\
&= \sin^2(x)\left(1 - \sin^2(y)\right) - \left(1 - \sin^2(x)\right)\sin^2(y) \\
&= \sin^2(x) - \sin^2(x)\sin^2(y) - \sin^2(y) + \sin^2(x)\sin^2(y) \\
&= \sin^2(x) - \sin^2(y)
\end{aligned}$$

□

Exercise 19 Prove that $\tan(x) + \tan(y) = \frac{\sin(x+y)}{\cos(x)\cos(y)}$.

Solution.

$$\begin{aligned}\tan(x) + \tan(y) &= \frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)} \\ &= \frac{\sin(x)\cos(y) + \sin(y)\cos(x)}{\cos(x)\cos(y)} \\ &= \frac{\sin(x+y)}{\cos(x)\cos(y)}\end{aligned}$$

□

Exercise 20 Use the law of exponents to rewrite and simplify the expressions:

(1) $x\left(3x^2\right)^3$.

(2) $\frac{x^{2n}\cdot x^{3n-1}}{x^{n+2}}$.

(3) $\frac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$.

Solution. (1) $27x^7$.

(2) x^{4n-3}.

(3) $a^{1/6}b^{-1/12}$.

□

Exercise 21 Given the following formula, determine whether they are one-to-one.

$$g(x) = \sqrt[3]{x}$$

Solution. It passes the horizontal line test so it is one-to-one. □

Exercise 22 Find a formula for the inverse of the following function

$$f(x) = \frac{4x-1}{2x+3}$$

Solution.

$$\begin{aligned}y = \frac{4x-1}{2x+3} &\Rightarrow y(2x+3) = 4x-1 \\ &\Rightarrow 2xy + 3y = 4x - 1 \\ &\Rightarrow 3y + 1 = 4x - 2xy \\ &\Rightarrow 3y + 1 = (4-2)x \\ &\Rightarrow x = \frac{3y+1}{4-2y} \\ &\Rightarrow f^{-1}(x) = \frac{3x+1}{4-2x}\end{aligned}$$

□