Homework 11

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An exercise marked with the symbol $\star$ is considered more difficult and will not be an exam question.

Exercise 1 State the terms of the following sums.

(1) $\sum_{j=2}^{5} 4\,j^3 - 1$

(2) $\sum_{k=1}^{4} (-2)^k - \frac{1}{2^k}$

(3) $\sum_{i=2}^{7} (-3)^{i-4}$

(4) $\sum_{i=1}^{3} f(x_i)^i$

Solution. (1) $31 + 107 + 255 + 499$

(2) $-\frac{5}{2} + \frac{15}{4} + -\frac{65}{8} + \frac{255}{16}$

(3) $\frac{1}{9} + -\frac{1}{3} + 1 + -3 + 9 + -27$

(4) $f(x_1) + f(x_2)^2 + f(x_3)^3$

□

Exercise 2 Use the $\sum$ symbol to represent the following sums.

(1) $2 + 4 + 6 + 8$

(2) $-\frac{1}{2} + \frac{1}{4} + -\frac{1}{6} + \frac{1}{8} + -\frac{1}{10}$

(3)$\star$ $-1 + 2 + -3 + 4 + -5 + 6 + -7$

(4) $f(\frac{x_1}{1}) - f(\frac{x_2}{2}) + f(\frac{x_3}{3}) - f(\frac{x_4}{4})$

Solution. (1) $\sum_{i=1}^{4} 2\,i$

(2) $\sum_{i=1}^{5} \frac{(-1)^i}{2\,i}$

(3) $\sum_{i=1}^{7} (-1)^i\, i$

(4) $\sum_{i=1}^{4} (-1)^{i+1} f\left(\frac{x_i}{i}\right)$

□

Exercise 3 Evaluate the following sums using the formulas from class.

(1) $\sum_{i=5}^{75} i$

(2) $\sum_{i=1}^{20} \frac{3}{2}\,i - \frac{5}{2}$

(3) $\sum_{i=1}^{25} (2\,i - 3)^2$

(4)⋆ $\sum_{i=1}^{25} 5\,\left(\frac{1}{4}\right)^i + \frac{1}{2}$

(4) $\sum_{i=6}^{72} \frac{i+2}{3} - \frac{i+3}{3}$

Solution. (1) 2840

(2) 265

(3) 18425

(4) $\frac{15950248680270505}{1125899906842624}$

(5) $-\frac{67}{3}$

□

Exercise 4 Use the formulas from class to express the following sums as functions of n.

(1) $\sum_{i=1}^{n-1} i$

(2) $\sum_{i=1}^{n-1} \frac{3\,i^2}{5\,n}$

(3) $\sum_{i=1}^{n-1} 6\,i^2 - 2\,i$

(4)⋆ $\sum_{i=10}^{2n} i^2 - 3\,i$

Solution. (1) $\frac{1}{2}\,n^2 - \frac{1}{2}\,n$

(2) $\frac{1}{5}\,n^2 - \frac{3}{10}\,n + \frac{1}{10}$

(3) $2\,n^3 - 4\,n^2 + 2\,n$

(4) $\frac{8}{3}\,n^3 - 4\,n^2 - \frac{8}{3}\,n - 150$

□

Exercise 5 Show that.

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Solution. Rewrite the sum as:

$$\sum_{i=1}^{n} (i^4 - (i-1)^4) = n^4 - 0^4$$

Also,

$$\begin{aligned}\sum_{i=1}^{n}\left(i^4-(i-1)^4\right) &= \sum_{i=1}^{n}\left(i^4-(i^4-4i^3+6i^2-4i+1)\right)\\ &= \sum_{i=1}^{n}4i^3-6i^2+4i-1\\ &= 4\sum_{i=1}^{n}i^3-6\frac{n(n+1)(2n+1)}{6}+4\frac{n(n+1)}{2}-n\\ &= 4\sum_{i=1}^{n}i^3-(2n^3+3n^2+n)+2(n^2+n)-n\\ &= 4(\sum_{i=1}^{n}i^3)-2n^3-3n^2-n+2n^2+2n-n\\ &= 4(\sum_{i=1}^{n}i^3-2n^3-n^2\end{aligned}$$

Then:

$$\begin{aligned}n^4 &= 4\sum_{i=1}^{n}i^3-2n^3-n^2\\ n^4+2n^3+n^2 &= 4\sum_{i=1}^{n}i^3\\ \sum_{i=1}^{n}i^3 &= \frac{n^2(n^2+2n+1)}{4}\\ &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

□

Exercise 6 ($\star$) Show that

$$\sum_{k=0}^{n-1}ar^k = a\frac{1-r^n}{1-r}$$

Solution.

$$\begin{aligned}\sum_{k=0}^{n-1} ar^k &= a + a\sum_{k=1}^{n} nr^k - ar^n \\ &= a + a\frac{r^{n+1}-r}{r-1} - ar^n \\ &= a\frac{1(r-1) + (r^{n+1}-r) - r^n(r-1)}{r-1} \\ &= a\frac{r-1+r^{n+1}-r-r^{n+1}+r^n}{r-1} \\ &= a\frac{-1+r^n}{r-1} \\ &= a\frac{1-r^n}{1-r}\end{aligned}$$

□

Exercise 7 For each of the following intervals, evaluate the length of Δx for each of the subintervals if we separate the interval into n equal parts. Give each interval.

(1) $[0,1]$, $n = 5$

(2) $[-2, \frac{3}{2}]$, $n = 10$

(3) $[1,5]$, $n = 12$

Solution. (1) $\frac{1}{5}$,

$$\left[0, \frac{1}{5}\right], \left[\frac{1}{5}, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, 1\right]$$

(2) $\frac{7}{20}$,

$$\left[-2, \frac{-33}{20}\right], \left[\frac{-33}{20}, \frac{-26}{20}\right], \left[\frac{-26}{20}, \frac{-19}{20}\right], \left[\frac{-19}{20}, \frac{-12}{20}\right],$$
$$\left[\frac{-12}{20}, \frac{-5}{20}\right], \left[\frac{-5}{20}, \frac{2}{20}\right], \left[\frac{2}{20}, \frac{9}{20}\right],$$
$$\left[\frac{9}{20}, \frac{16}{20}\right], \left[\frac{16}{20}, \frac{23}{20}\right], \left[\frac{23}{20}, \frac{3}{2}\right]$$

(3) $\frac{1}{3}$,

$$\left[1, \frac{4}{3}\right], \left[\frac{4}{3}, \frac{5}{3}\right], \left[\frac{5}{3}, 2\right], \left[2, \frac{7}{3}\right],$$
$$\left[\frac{7}{3}, \frac{8}{3}\right], \left[\frac{8}{3}, 3\right], \left[3, \frac{10}{3}\right],$$
$$\left[\frac{10}{3}, \frac{11}{3}\right], \left[\frac{11}{3}, 4\right], \left[4, \frac{13}{3}\right],$$
$$\left[\frac{13}{3}, \frac{14}{3}\right], \left[\frac{14}{3}, 5\right]$$

□

Exercise 8 If $f(x)$ and $g(x)$ are two integrable functions on the given intervals, and knowing that

$$\int_3^6 f(x)\,dx = 12, \qquad \int_6^{10} f(x)\,dx = 1$$
$$\int_3^0 g(x)\,dx = -1, \qquad \int_0^6 g(x)\,dx = 15$$
$$\int_6^{10} g(x)\,dx = 11$$

evaluate the following definite integrals using the properties of definite integrals from class.

(1) $\int_3^{10} f(x)\,dx$

(2) $\int_{10}^{6} f(x)\,dx$

(3) $\int_6^6 f(x)\,dx$

(4) $\int_3^{10} (3f(x) + 2g(x))\ dx$

(5) $\int_3^6 (2f(x) + 4g(x))\ dx$

Solution. (1) $12 + 1 = 13$

(2) -1

(3) 0

(4) $3(12 + 1) + 2(11 + 15) = 36 + 52 = 88$

(5) $2 \cdot 12 + 4(15 - 1) = 24 + 56 = 80$

□