Tutorial 3

Question 1 The transition matrix T, from a basis $\mathcal{B}$ to a basis $\mathcal{C}$ is given by

$$T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Does $\mathcal{C}$ have the same or opposite orientation to basis $\mathcal{B}$?

Solution. The determinant of T is given by: $\det T = -1$. Since the determinant is negative, $\mathcal{B}$ and $\mathcal{C}$ have opposite orientation. □

Question 2 Let $(\mathbf{e}, \mathbf{f})$ be an orthonormal basis for $\mathbb{E}^2$. For each of the following linear operators in $\mathbb{E}^2$, determine if they are a rotation, reflection or neither.

$$\mathbf{e} \mapsto -\mathbf{e} \qquad \mathbf{e} \mapsto \mathbf{e} - \mathbf{f} \qquad \mathbf{e} \mapsto -\frac{1}{2}\mathbf{e} + \frac{\sqrt{3}}{2}\mathbf{f}$$

$$\mathbf{f} \mapsto -\mathbf{f} \qquad \mathbf{f} \mapsto 2\mathbf{e} + \mathbf{f} \qquad \mathbf{f} \mapsto \frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}$$

If they are a rotation or reflection, give their angle of rotation/reflection.

Solution. Orthogonal operators in $\mathbb{E}^2$ are rotations if and only if the orientation is preserved aand they are reflections if and only if the orientation is not preserved. Let's look at each case one by one:

(1) In the first case we end up with

$$[P]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The determinant of P is equal to 1 and therefore it is orthogonal, and is a rotation.

The standard matrix for a rotation is given by:

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

Therefore we want to find a φ such that $\cos\varphi = -1$ and $\sin\varphi = 0$. This is precisely when $\varphi = \pi$. Therefore $P = P_\pi$.

(2) In the second case we end up with

$$[Q]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

The determinant is equal to $(1)(1) - (2)(-1) = 3$ and therefore Q is not orthogonal (and therefore neither a reflection nor a rotation.

(3) In the third case we end up with

$$[R]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

The determinant is equal to $-\frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{1}{4} - \frac{3}{4} = -1$. Therefore R is orthogonal and is a reflection.

The standard matrix for a rotation is given by:

$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}.$$

Therefore we want to find a φ such that $\cos\varphi = -\frac{1}{2}$ and $\sin\varphi = \frac{\sqrt{3}}{2}$. This is precisely when $\varphi = \frac{2\pi}{3}$. Therefore $R = Q_{\frac{2\pi}{3}}$.

□