Tutorial 2

Question 1 We continue Question 1 from Tutorial 1. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis for $\mathbb{E}^3$. Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ where

$$\begin{aligned}\mathbf{f}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{f}_2 &= \mathbf{e}_1 + \lambda\mathbf{e}_3, \quad \lambda \in \mathbb{R} \\ \mathbf{f}_3 &= -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3.\end{aligned}$$

Recall that

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & \lambda & -1 \end{bmatrix}$$

and that $\det {}_{\mathcal{B}}T_{\mathcal{C}} = 3 - 3\lambda$. Let $\lambda = 0$ and $\mathcal{B}$ be an orthonormal basis. Give three reasons $\mathcal{C}$ is not an orthonormal basis.

Solution. There are lots of ways to show $\mathcal{C}$ is not orthonormal, here are a selection:

- $\mathcal{C}$ is orthonormal if and only if ${}_{\mathcal{B}}T_{\mathcal{C}}$ is orthogonal. A quick check shows $({}_{\mathcal{B}}T_{\mathcal{C}})^{\mathsf{T}}\,{}_{\mathcal{B}}T_{\mathcal{C}} \neq I_3$.
- Orthogonal matrices have determinant ± 1, however from the previous question we know $\det({}_{\mathcal{B}}T_{\mathcal{C}}) = 3$ and so cannot be orthogonal.
- Alternatively, we can work directly with $\mathcal{C}$ via inner products. Consider $\langle \mathbf{f}_1,\, \mathbf{f}_2 \rangle$:

$$\begin{aligned}\langle \mathbf{f}_1,\, \mathbf{f}_2 \rangle &= \langle \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,\, \mathbf{e}_1 \rangle \\ &= \langle \mathbf{e}_1,\, \mathbf{e}_1 \rangle + \langle \mathbf{e}_2,\, \mathbf{e}_1 \rangle + \langle \mathbf{e}_3,\, \mathbf{e}_1 \rangle \\ &= 1\end{aligned}$$

  but $\langle \mathbf{f}_1,\, \mathbf{f}_2 \rangle$ must equal 0 for $\mathcal{C}$ to be orthogonal. We could take the inner product of almost any pair of vectors to find this contradiction.

□

Question 2 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis for $\mathbb{E}^3$. Let $P_\lambda \colon \mathbb{E}^3 \to \mathbb{E}^3$ be the linear operator that maps

$$\mathbf{e}_1 \mapsto -\mathbf{e}_1; \qquad \mathbf{e}_2 \mapsto \mathbf{e}_2 + (1+\lambda)\mathbf{e}_3; \qquad \text{and} \qquad \mathbf{e}_3 \mapsto -2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3.$$

(1) Write down the matrix of the linear operator P_λ with respect to the $\mathcal{B}$ basis.

(2) Calculate the determinant and the trace of P_λ.

Solution. (1) The matrix of the linear operator P_λ with respect to the basis $\mathcal{B}$ is given by:

$$[P_\lambda]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & (1+\lambda) & 1 \end{bmatrix}$$

(2) The determinant and trace are obtained from $[P_\lambda]_{\mathcal{B}}$:

$$\det\,[P_\lambda]_{\mathcal{B}} = (-1)(1)(1) - (-1)(-1)(1+\lambda) = -2 - \lambda \quad \operatorname{Tr}\,[P_\lambda]_{\mathcal{B}} = -1 + 1 + 1 = 1$$

□

Question 3 Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be another basis for $\mathbb{E}^3$. Let R_μ be the linear operator that maps

$$\mathbf{f}_1 \mapsto -\mathbf{f}_1; \qquad \mathbf{f}_2 \mapsto \mu\mathbf{f}_1 + \mathbf{f}_3; \qquad \text{and} \qquad \mathbf{f}_3 \mapsto -\mathbf{f}_1 + \mathbf{f}_2 + (\mu^2+1)\mathbf{f}_3.$$

(1) Write down the matrix of the linear operator R_μ in the $\mathcal{C}$ basis.

(2) Calculate the determinant and the trace of R_μ.

Solution. (1) The matrix of the linear operator R_μ in the $\mathcal{C}$ basis is given by:

$$[R_\mu]_\mathcal{C} = \begin{bmatrix} -1 & \mu & -1 \\ 0 & 0 & 1 \\ 0 & 1 & (\mu^2+1) \end{bmatrix}$$

(2) The determinant and trace are obtained from $[R_\mu]_\mathcal{C}$:

$$\det{[R_\mu]_\mathcal{C}} = -(-1)(1)(1) = 1 \quad \operatorname{Tr}{[R_\mu]_\mathcal{C}} = -1 + 0 + \mu^2 + 1 = \mu^2$$

□

Question 4 There exists a pair of real numbers λ and $\mu \in \mathbb{R}$ such that P_λ and R_μ are the same linear operator.

(1) Using the answers you have calculated so far, what are the possible values of λ and μ?

Hint: *You should find a single value for λ and two possibilities for μ.*

Solution. Since we are told that P_λ and R_μ are the same linear operator, the determinant and trace should match up. In other words:

$$\det P_\lambda = \det R_\mu \Rightarrow -2 - \lambda = 1 \Rightarrow \lambda = -3$$
$$\operatorname{Tr} P_\lambda = \operatorname{Tr} R_\mu \Rightarrow \mu^2 = 1 \Rightarrow \mu = \pm 1$$

□

Question 5 The transition matrix T, from basis $\mathcal{B}$ to basis $\mathcal{C}$; and its inverse are given by

$$T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(1) Using the value of λ you calculated in Question 4, calculate the matrix for the linear operator P_λ in the $\mathcal{C}$ basis; and deduce the correct value of μ.

(2) Using the value of λ you calculated in Question 4, can you specify a matrix with the same determinant and trace as P_λ, but that is not the matrix of P_λ with respect to any orthonormal basis?

Solution. (1) We know that $T = {}_\mathcal{B}T_\mathcal{C}$. In other words $T^{-1} = {}_\mathcal{C}T_\mathcal{B}$ and by Proposition 1.43 we have the following:

$$\begin{aligned} [P_{-3}]_\mathcal{C} &= {}_\mathcal{C}T_\mathcal{B}[P_{-3}]_\mathcal{B}\,{}_\mathcal{B}T_\mathcal{C} \\ &= \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= [R_{-1}]_\mathcal{C} \end{aligned}$$

Therefore $\mu = -1$.

(2) Note that P_{-3} is not orthogonal since:

$$[P_{-3}]_\mathcal{B}^\mathsf{T}[P_{-3}]_\mathcal{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -3 \\ 2 & -3 & 6 \end{bmatrix} \neq I_3$$

However we can find an operator Q such that:

$$\det Q = 1 \quad \operatorname{Tr} Q = 1 \quad Q \text{ orthogonal}$$

An example is:

$$[Q]_\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

□