Tutorial 1

Question 1 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis for $\mathbb{E}^3$. Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ where

$$\begin{aligned}\mathbf{f}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\\ \mathbf{f}_2 &= \mathbf{e}_1 + \lambda\mathbf{e}_3, \quad \lambda \in \mathbb{R}\\ \mathbf{f}_3 &= -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3\end{aligned}$$

(1) For what values of λ is $\mathcal{C}$ a basis?

(2) When $\mathcal{C}$ is not a basis, find a linear dependence between $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$.

Solution. (1) The transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & \lambda & -1 \end{bmatrix}.$$

$\mathcal{C}$ is a basis if and only if:

$$\det(_{\mathcal{B}}T_{\mathcal{C}}) = 3 - 3\lambda \neq 0,$$

therefore $\mathcal{C}$ is a basis for $\lambda \neq 1$.

(2) We know $\mathcal{C}$ is not a basis for $\lambda = 1$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be scalars such that

$$\begin{aligned}\mathbf{0} &= \alpha_1\mathbf{f}_1 + \alpha_2\mathbf{f}_2 + \alpha_3\mathbf{f}_3\\ &= \alpha_1(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + \alpha_2(\mathbf{e}_1 + \mathbf{e}_3) + \alpha_3(-\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3)\\ &= (\alpha_1 + \alpha_2 - \alpha_3)\mathbf{e}_1 + (\alpha_1 + 2\alpha_3)\mathbf{e}_2 + (\alpha_1 + \alpha_2 - \alpha_3)\mathbf{e}_3.\end{aligned}$$

This gives rise to the pair of simultaneous equations:

$$\begin{aligned}\alpha_1 + \alpha_2 - \alpha_3 &= 0\\ \alpha_1 + 2\alpha_3 &= 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}\alpha_1 &= -2\alpha_3\\ \alpha_2 &= 3\alpha_3\end{aligned}$$

Setting $\alpha_3 = 1$ gives a linear dependence $-2\mathbf{f}_1 + 3\mathbf{f}_2 + \mathbf{f}_3 = \mathbf{0}$.

$\square$

Question 2 (1) Which of the following are inner products on $\mathbb{E}^2$? [where $\mathbf{x} = [x_1, x_2]^\mathsf{T}$, $\mathbf{y} = [y_1, y_2]^\mathsf{T}$ in some basis.]

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \qquad \langle \mathbf{x}, \mathbf{y} \rangle = x_1 + y_2$$
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1x_2 + y_1y_2 \qquad \langle \mathbf{x}, \mathbf{y} \rangle = x_1y_2 + x_2y_1$$

(2) (*) Define an inner product on a basis of $\mathbb{E}^2$ by

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = a, \ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle = b, \ \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = c, \quad a, b, c \in \mathbb{R}$$

and extending linearly to all of $\mathbb{E}^2$. What conditions must a, b, c satisfy for this to define an inner product?

The following identity may be of use:

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = \frac{(ax_1 + bx_2)^2 + (ac - b^2)x_2^2}{a}$$

Solution. (1) None of them are: the easiest way to break most of these is the positive definite axiom!

- $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ is not positive-definite: every vector $\mathbf{x}$ satisfies $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ rather than just the zero vector.
- $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 + y_2$ is not symmetric, and you can break the positive-definite condition with $\mathbf{x} = [-1, -1]^\mathsf{T}$.

- $\langle \mathbf{x},\, \mathbf{y}\rangle = x_1x_2 + y_1y_2$ is not linear: consider

$$\begin{aligned}\langle \lambda\mathbf{x},\, \mathbf{x}\rangle = (\lambda x_1)(\lambda x_2) + x_1x_2 = (\lambda^2+1)(x_1x_2) &= \frac{\lambda^2+1}{2}\langle \mathbf{x},\, \mathbf{x}\rangle \\ &\neq \lambda\langle \mathbf{x},\, \mathbf{x}\rangle.\end{aligned}$$

  You can break the positive-definite condition with $\mathbf{x} = [1, -1]^\mathsf{T}$.

- $\langle \mathbf{x},\, \mathbf{y}\rangle = x_1y_2 + x_2y_1$ can also be broken by $\mathbf{x} = [1, -1]^\mathsf{T}$ (although it does at least satisfy symmetry and linearity).

(2) We first calculate the inner product of generic vectors $\mathbf{x}, \mathbf{y}$:

$$\begin{aligned}\langle \mathbf{x},\, \mathbf{y}\rangle &= \langle x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\, y_1\mathbf{e}_1 + y_2\mathbf{e}_2\rangle \\ &= x_1y_1\langle \mathbf{e}_1,\, \mathbf{e}_1\rangle + x_1y_2\langle \mathbf{e}_1,\, \mathbf{e}_2\rangle + x_2y_1\langle \mathbf{e}_2,\, \mathbf{e}_1\rangle + x_2y_2\langle \mathbf{e}_2,\, \mathbf{e}_2\rangle \\ &= ax_1y_1 + b(x_1y_2 + x_2y_1) + cx_2y_2.\end{aligned}$$

As we defined this inner product on the basis and extended linearly, it automatically satisfies linearity. It is also quick to check it satisfies symmetry, only positive-definite remains.

The first thing to note is that $a, c > 0$ as $\langle \mathbf{e}_1,\, \mathbf{e}_1\rangle = a > 0$ by positive-definiteness, c shown similarly. For a generic vector $\mathbf{x}$:

$$\begin{aligned}\langle \mathbf{x},\, \mathbf{x}\rangle &= ax_1^2 + 2bx_1x_2 + cx_2^2 \\ &= \frac{(ax_1 + bx_2)^2 + (ac - b^2)x_2^2}{a}.\end{aligned}$$

Clearly $\langle \mathbf{0},\, \mathbf{0}\rangle = 0$, we need to ensure the numerator is positive for all $\mathbf{x} \neq \mathbf{0}$. Note that $(ax_1+bx_2)^2$ is always non-negative, but may be zero. $(ac-b^2)x_2^2$ is positive if and only if $ac-b^2 > 0$. Therefore the conditions

$$a, c > 0,\; ac - b^2 > 0$$

are enough to ensure $\langle -,\, -\rangle$ defines an inner product.

$\square$