Tutorial 2

Question 1 We continue Question 1 from Tutorial 1. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis for $\mathbb{E}^3$. Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ where

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{f}_2 &= \mathbf{e}_1 + \lambda\mathbf{e}_3, \quad \lambda \in \mathbb{R} \\ \mathbf{f}_3 &= -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3. \end{aligned}$$

Recall that

$$_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & \lambda & -1 \end{bmatrix}$$

and that $\det {_{\mathcal{B}}T_{\mathcal{C}}} = 3 - 3\lambda$. Let $\lambda = 0$ and $\mathcal{B}$ be an orthonormal basis. Give three reasons $\mathcal{C}$ is not an orthonormal basis.

Question 2 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis for $\mathbb{E}^3$. Let $P_\lambda \colon \mathbb{E}^3 \to \mathbb{E}^3$ be the linear operator that maps

$$\mathbf{e}_1 \mapsto -\mathbf{e}_1; \qquad \mathbf{e}_2 \mapsto \mathbf{e}_2 + (1+\lambda)\mathbf{e}_3; \qquad \text{and} \qquad \mathbf{e}_3 \mapsto -2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3.$$

(1) Write down the matrix of the linear operator P_λ with respect to the $\mathcal{B}$ basis.

(2) Calculate the determinant and the trace of P_λ.

Question 3 Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be another basis for $\mathbb{E}^3$. Let R_μ be the linear operator that maps

$$\mathbf{f}_1 \mapsto -\mathbf{f}_1; \qquad \mathbf{f}_2 \mapsto \mu\mathbf{f}_1 + \mathbf{f}_3; \qquad \text{and} \qquad \mathbf{f}_3 \mapsto -\mathbf{f}_1 + \mathbf{f}_2 + (\mu^2+1)\mathbf{f}_3.$$

(1) Write down the matrix of the linear operator R_μ in the $\mathcal{C}$ basis.

(2) Calculate the determinant and the trace of R_μ.

Question 4 There exists a pair of real numbers λ and $\mu \in \mathbb{R}$ such that P_λ and R_μ are the same linear operator.

(1) Using the answers you have calculated so far, what are the possible values of λ and μ?

Hint: *You should find a single value for λ and two possibilities for μ.*

Question 5 The transition matrix T, from basis $\mathcal{B}$ to basis $\mathcal{C}$; and its inverse are given by

$$T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(1) Using the value of λ you calculated in Question 4, calculate the matrix for the linear operator P_λ in the $\mathcal{C}$ basis; and deduce the correct value of μ.

(2) Using the value of λ you calculated in Question 4, can you specify a matrix with the same determinant and trace as P_λ, but that is not the matrix of P_λ with respect to any orthonormal basis?