Week 8

2.5 Integration of 1-forms over curves

In Subsubsection 2.3.2, we showed how 1-forms could arise as the differential of a 0-form. Intuitively, we should also have an inverse operation that allows us to integrate a 1-form. This is precisely what we consider in this section.

We shall only consider integration of 1-forms over curves. The reason for this is a 1-form ω requires a point $\mathbf{P}$ and a tangent vector $\mathbf{v}$ to give back a real number $\omega(\mathbf{P}, \mathbf{v})$. Curves with a parametrisation have all this information, as each point has a velocity vector already associated to it.

Let $\omega = g_1 \mathrm{d}x_1 + \cdots + g_n \mathrm{d}x_n$ be a 1-form and $\boldsymbol{\gamma} : [a, b] \to \mathbb{A}^n$ a parametrisation of a curve C. We can apply ω to the parametrisation $\boldsymbol{\gamma}$ by evaluating at the point $\boldsymbol{\gamma}(t)$ with velocity vector $\boldsymbol{\gamma}'(t)$:

$$\begin{aligned}\omega(\boldsymbol{\gamma}(t), \boldsymbol{\gamma}'(t)) &= \sum_{i=1}^{n} g_i(\boldsymbol{\gamma}(t)) \mathrm{d}x_i(\boldsymbol{\gamma}'(t)) \\ &= \sum_{i=1}^{n} g_i(\boldsymbol{\gamma}(t)) \gamma_i'(t).\end{aligned}$$

Note that if $\omega = \mathrm{d}f$ is exact, this is the directional derivative of f at $\boldsymbol{\gamma}(t)$ in the direction of the velocity vector $\boldsymbol{\gamma}'(t)$.

Definition 2.53 (Integration of 1-form over curves) Let $\omega = g_1 \mathrm{d}x_1 + \cdots + g_n \mathrm{d}x_n$ be a 1-form and $\boldsymbol{\gamma} \colon [a, b] \to \mathbb{A}^n$ a smooth, regular parametrisation of C. The *integral of ω over C* is

Wikipedia: integral of ω over C

(2.21)

Remark 2.54 The same definition holds if our curve is defined over an open interval (a, b), as removing a finite number of points from the curve does not impact the integral.

Example 2.55 Consider the 1-form $\omega = x\mathrm{d}x + y\mathrm{d}y$ and the curve

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \;\middle|\; y = x^2 \,,\, 0 < x < 1 \right\}.$$

Compute the integral $\int_C \omega$.

We have computed lots of parametrisations of C, so a natural question is how does this integral change if we change the parametrisation? The following theorem shows that if we fix the orientation of C, the value of the integral does not change.

Theorem 2.56 *Let C be an oriented curve. The integral $\int_C \omega$ does not depend on the parametrisation of C.*

Let C' be the curve C with the opposite orientation. Then

Example 2.57 The first part of Theorem 2.56 states the value of $\int_C \omega$ does not change if we reparametrise C while preserving orientation. We shall verify this for the curve C and the 1-form ω from Example 2.55.

We consider another parametrisation of C,

$$\begin{aligned} \gamma_3\colon \left(0, \frac{\pi}{2}\right) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} \sin t \\ \sin^2 t \end{pmatrix} \end{aligned} \quad , \quad \gamma_3'(t) = \begin{bmatrix} \cos t \\ 2 \sin t \cos t \end{bmatrix}.$$

Do we get the same value of $\int_C \omega$?

Example 2.58 The second part of Theorem 2.56 states that if we consider the integral C with the opposite orientation, we just need to flip the sign on the value of the integral. Again, we shall verify this for the curve C and the 1-form ω from Example 2.55.

Consider another parametrisation of C:

$$\begin{aligned} \boldsymbol{\gamma}_4 \colon (0,1) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} 1-t \\ (1-t)^2 \end{pmatrix}, \quad \boldsymbol{\gamma}_4'(t) = \begin{bmatrix} -1 \\ 2t-2 \end{bmatrix}. \end{aligned}$$

Do we get the opposite value of $\int_C \omega$?

2.6 Integrating exact 1-forms

When we defined exact 1-forms, we noted that they would be particularly nice when it came to integrating them. The following theorem is the reason for this:

Theorem 2.59 *Let $\omega = \mathrm{d}f$ be an exact 1-form and C a curve in $\mathbb{A}^n$. The integral of ω over C depends only on the value of f at the endpoints of C.*

Explicitly, for a parametrisation $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{A}^n$ *of* C*, the integral of* ω *over* C

(2.22)

Proof. Recall that is $\omega = \mathrm{d}f$ is exact, then $\mathrm{d}f(\mathbf{P},\mathbf{v})$ is just the directional derivative of f at $\mathbf{P}$ along $\mathbf{v}$. In particular, $\mathrm{d}f(\boldsymbol{\gamma}(t),\boldsymbol{\gamma}'(t))$ is the directional derivative along f along $\boldsymbol{\gamma}$. Therefore

$$\int_C \mathrm{d}f = \int_a^b \mathrm{d}f(\boldsymbol{\gamma}(t),\boldsymbol{\gamma}'(t))\mathrm{d}t = \int_a^b D_{\boldsymbol{\gamma}'} f \mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(\boldsymbol{\gamma}(t)))\mathrm{d}t$$
$$= \int_a^b f(\boldsymbol{\gamma}(t)) = [f(\boldsymbol{\gamma}(t))]_a^b = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)).$$

□

Example 2.60 If we reconsider the 1-form from Example 2.55, we have already seen that this is an exact 1-form whose corresponding 0-form is $f = \frac{x^2+y^2}{2}$. Considering the curve C from the same example, what is $\int_C \omega$?

Remark 2.61 As the integral of exact 1-forms depends only on the endpoints, we do not care what path the curve takes between those endpoints, it will have no impact on the value of the integral.

Wikipedia: closed

We call a curve *closed* if its endpoints are the same point. For example, circles and ellipses are both closed curves, as are any deformations of them. Exact forms are even easier to integrate over closed curves.

Corollary 2.62 *If* C *is a closed curve and* ω *an exact* 1*-form, then* $\int_C \omega = \_\_$.

Proof. As C is closed, its endpoints $\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)$ are equal and so $f(\boldsymbol{\gamma}(a)) = f(\boldsymbol{\gamma}(b))$ for any 0-form f. Using this and Theorem 2.59, we have

$$\int_C \omega = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)) = 0.$$

□

3 Conic sections

In this section we consider very important curves that you have no doubt encountered before: ellipses; hyperbolas; and parabolas. We will begin with the geometric definitions of these curves, we will follow this with the algebraic definitions in terms of both Cartesian and polar coordinates. These curves can all be realised as sections of a cone and are therefore collectively known by the name *conic sections*, see Figure 17. We will revisit conic sections again in Section 5 when we review projective geometry, giving a more modern perspective on these curves.

3.1 Geometric definitions

Geometrically we define an ellipse or hyperbola, as the locus of points that satisfy some geometric condition with respect to a pair of points called the *foci* (or sometimes *focuses*) of the curve. Similarly, we define a parabola as the locus of points that satisfy a geometric condition with respect to a single focus and a line called the *directrix*.

The following geometric definitions take place in affine space. Recall that for a point $\mathbf{P}$ in $\mathbb{A}^n$ and a vector $\mathbf{v} \in \mathbb{E}^n$ we can add the vector to the point to get a new point in affine space: $\mathbf{P} + \mathbf{v} \in \mathbb{A}^n$. In fact, for each pair of points $\mathbf{P}, \mathbf{Q} \in \mathbb{A}^n$ there is a unique vector $\mathbf{v} \in \mathbb{E}^n$ such that $\mathbf{P} + \mathbf{v} = \mathbf{Q}$. Thus, although there is no addition of points in affine space, there is a concept of subtraction. The notation $\mathbf{Q} - \mathbf{P}$, is really a shorthand meaning: "the vector $\mathbf{v}$ for which $\mathbf{P} + \mathbf{v} = \mathbf{Q}$".

The definitions that follow involve distances between points in affine space. As there is a unique vector from $\mathbf{P}$ to $\mathbf{Q}$, we use this vector to define the distance between points. Using the idea of subtraction of points above, we can denote the distance between $\mathbf{P}$ and $\mathbf{Q}$ as $\|\mathbf{Q} - \mathbf{P}\| = \|\mathbf{P} - \mathbf{Q}\|$.

Definition 3.1 (Ellipse in the affine plane, see Figure 8) Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two points, called *foci*, in the affine plane $\mathbb{A}^2$ and let $c \in \mathbb{R}$ be half the distance between the two foci: $\|\mathbf{F}_2 - \mathbf{F}_1\| = 2c$.

Wikipedia: foci

Then for each constant $a > c \geq 0$ we define the *ellipse* with foci $\mathbf{F}_1$, $\mathbf{F}_2$ and with distance $2a$ to be the set of points $\mathbf{P}$, such that the sum of the distances of $\mathbf{P}$ to each focus is $2a$:

Wikipedia: ellipse

$$\text{Ellipse} = \tag{3.1}$$

If $\mathbf{F}_1$ and $\mathbf{F}_2$ are the same point then this is simply the circle with centre $\mathbf{F}_1$ and radius a.

Remark 3.2 Notice that if we allowed $a = c$, then this definition would degenerate to the line segment from $\mathbf{F}_1$ and $\mathbf{F}_2$; whilst with $a < c$, no

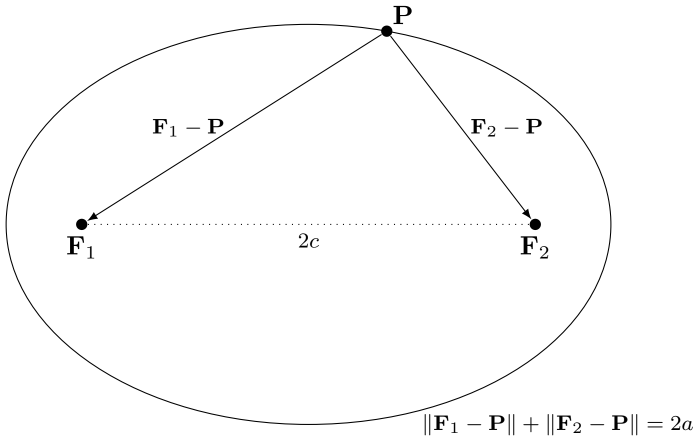

Figure 8: An ellipse with foci $\mathbf{F}_1$ and $\mathbf{F}_2$ and distance $2a$.

points would satisfy the condition.

Definition 3.3 (Hyperbola in the affine plane, see Figure 9) Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two points in the affine plane $\mathbb{A}^2$, and let $c \in \mathbb{R}$ be half the distance between the two foci.

Wikipedia: hyperbola

Then for each constant a, with $0 < a < c$, we define the *hyperbola* with foci $\mathbf{F}_1$, $\mathbf{F}_2$ and with distance $2a$ to be the set of points $\mathbf{P}$, such that the absolute difference between the distances of $\mathbf{P}$ to each focus is $2a$:

$$\text{Hyperbola} = \tag{3.2}$$

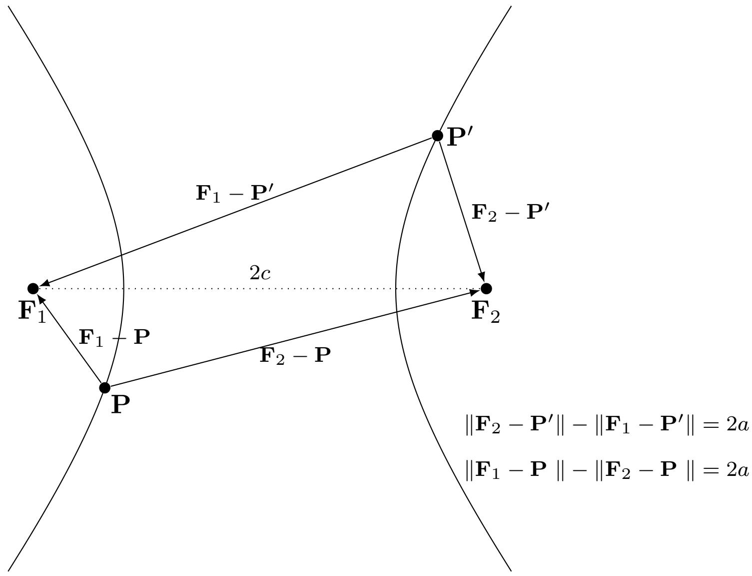

Figure 9: A hyperbola with foci $\mathbf{F}_1$ and $\mathbf{F}_2$ and distance $2a$.

Remark 3.4 Notice that if we allowed $a = c$ then this would degenerate to the union of two half-lines: the first beginning at $\mathbf{F}_1$ and extending infinitely away from $\mathbf{F}_2$; the second beginning at $\mathbf{F}_2$ and extending infinitely away from $\mathbf{F}_1$. On the other hand, if we allowed $a = 0$ then this would degenerate to the perpendicular bisector of the line from $\mathbf{F}_1$ to $\mathbf{F}_2$.

Definition 3.5 (Parabola in the affine plane, see Figure 10) Let $\mathbf{F}$ be a point, called a focus, in the affine plane $\mathbb{A}^2$, and let l be a line, called the *directrix*.

Then we define the *parabola* with focus $\mathbf{F}$ and directrix l to be set of points $\mathbf{P}$, such that the distance between $\mathbf{P}$ and $\mathbf{F}$ is equal to the distance between $\mathbf{P}$ and l. Denoting the closest point on the line l to the point $\mathbf{P}$ by $l_\mathbf{P}$ we have

Wikipedia: parabola

$$\text{Parabola} = \tag{3.3}$$

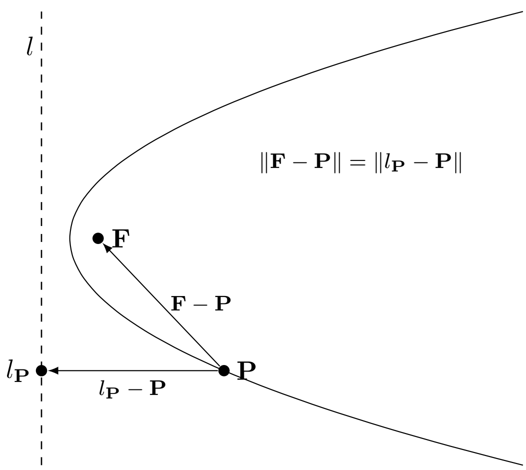

Figure 10: A parabola with focus $\mathbf{F}$ and directrix l.

3.2 Algebraic definitions

In this section we will give algebraic expressions for ellipses, hyperbolas and parabolas. In order to give these expressions we must use a coordinate system for affine space; initially we will restrict ourselves to Cartesian coordinates. Recall from Example 2.4 that we can specify a Cartesian coordinate system by selecting a point $\mathbf{O} \in \mathbb{A}^n$, called the origin, and selecting an orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ for $\mathbb{E}^n$.

3.2.1 Expression for an ellipse

We shall first derive an equation for the ellipse. Let C be an ellipse with foci $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{A}^2$ and distance $2a \in \mathbb{R}$. We first need to define a Cartesian coordinate system: $\mathbf{O}, (\mathbf{e}_x, \mathbf{e}_y)$ (see Figure 11)

- Let the origin be the point half way between $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$\mathbf{0} =$$

- Let the first basis vector be a unit vector in the same direction as the vector from $\mathbf{F}_1$ to $\mathbf{F}_2$:

$$\mathbf{e}_x =$$

- Let $\mathbf{e}_y$ be either of the unit vectors __________________ to $\mathbf{e}_x$.

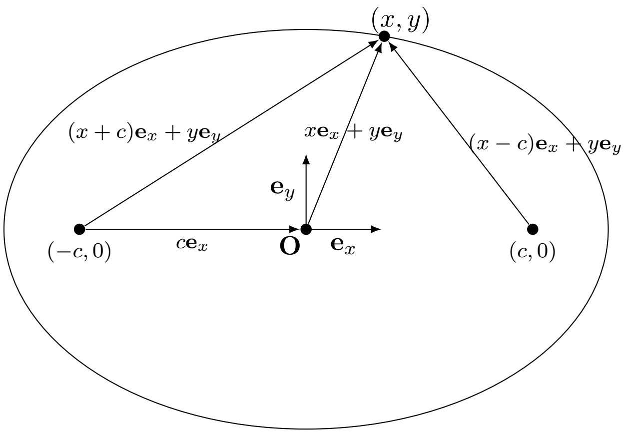

Figure 11: An ellipse shown against a Cartesian coordinate system.

Letting $2c$ be the distance between the two foci, we can express the $\mathbf{F}_1$ and $\mathbf{F}_2$ using this coordinate system:

$$\mathbf{F}_1 = (-c, 0) \qquad \mathbf{F}_2 = (c, 0).$$

Let $\mathbf{P} = (x, y)$ be a point on the ellipse. Then by definition

$$\begin{aligned} 2a &= \|\mathbf{P} - \mathbf{F}_1\| + \|\mathbf{P} - \mathbf{F}_2\| \\ &= \\ &= \end{aligned}$$

We have proved that every point on the ellipse satisfies the equation

$$. \tag{3.4}$$

We now show the converse: that any point satisfying equation (3.4) lies on the ellipse.

Let $\mathbf{Q}$ be the point with coordinates (x, y) and assume that these values satisfy the identity of equation (3.4). Then

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right).$$

We wish to calculate the distances between $\mathbf{Q}$ and the foci $\mathbf{F}_1 = (-c, 0)$ and $\mathbf{F}_2 = (c, 0)$. Recall that $a^2 = b^2 + c^2$.

Now since $0 \leq c < a$ and $|x| < a$ we see that

$$\|\mathbf{Q} - \mathbf{F}_1\| = \tag{3.5}$$

A similar process for the distance to $\mathbf{F}_2$ shows that

Putting this together we see that

$$\|\mathbf{Q} - \mathbf{F}_1\| + \|\mathbf{Q} - \mathbf{F}_2\| =$$

Using the analysis above we come to the algebraic definition of an ellipse.

Proposition 3.6 (Equation of an ellipse) *Let C be a curve in $\mathbb{A}^2$. The curve C is an ellipse if and only if there exists a Cartesian coordinate system (x, y) of $\mathbb{A}^2$ and real numbers $a \geq b > 0$ such that*

$$C = \left\{ (x, y) \in \mathbb{A}^2 \;\middle|\; \qquad \right\}. \tag{3.6}$$

Moreover, $a \in \mathbb{R}$ is exactly as in Definition 3.1; and if $2c$ is the distance between the foci then $b^2 = a^2 - c^2$.

Just as we derived an expression for an ellipse in terms of a Cartesian coordinate system we can do the same for hyperbolas and parabolas. We leave the required analysis as an exercise in each case and merely present the definitions.

Proposition 3.7 (Equation of a hyperbola) *Let C be a curve in $\mathbb{A}^2$. The curve C is a hyperbola if and only if there exists a Cartesian coordinate system (x, y) of $\mathbb{A}^2$ and real numbers $a > 0$, $b > 0$ such that*

$$C = \left\{ (x, y) \in \mathbb{A}^2 \;\middle|\; \qquad\qquad\qquad\qquad \right\}.$$

Moreover, $a \in \mathbb{R}$ is exactly as in Definition 3.3; and if $2c$ is the distance between the foci then $b^2 = c^2 - a^2$.

Proposition 3.8 (Equation of a parabola) *Let C be a curve in $\mathbb{A}^2$. The curve C is a parabola if and only if there exists a Cartesian coordinate system (x, y) of $\mathbb{A}^2$ and a real number $p > 0$ such that*

$$C = \left\{ (x, y) \in \mathbb{A}^2 \;\middle|\; \qquad\qquad\qquad \right\}.$$

Moreover, $p \in \mathbb{R}$ is the distance between the directrix and the focus.

It is useful to compile all the relevant details, which we do in the following table.

	Geometric	**Algebraic**	**Parameters**
Ellipse	$\lVert \mathbf{F}_1 - \mathbf{P} \rVert + \lVert \mathbf{F}_2 - \mathbf{P} \rVert = 2a$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$2c = \lVert \mathbf{F}_2 - \mathbf{F}_1 \rVert$; $b^2 = a^2 - c^2$
Hyperbola	$\lvert \lVert \mathbf{F}_1 - \mathbf{P} \rVert - \lVert \mathbf{F}_2 - \mathbf{P} \rVert \rvert = 2a$	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$2c = \lVert \mathbf{F}_2 - \mathbf{F}_1 \rVert$; $b^2 = c^2 - a^2$
Parabola	$\lVert \mathbf{F} - \mathbf{P} \rVert = \lVert l_{\mathbf{P}} - \mathbf{P} \rVert$	$y^2 = 2px$	$p = \lVert \mathbf{F} - l_{\mathbf{F}} \rVert$

Table 1: A comparison of the geometric and algebraic definitions of the ellipse, hyperbola and parabola.