Week 7

2.4 Curves in $\mathbb{A}^n$

2.4.1 Definitions

Aside from points, curves are the simplest geometric objects we can work with in affine space. This does not mean they are easy: even the simplest objects have some real complexity to them!

We shall define curves via parametrisation. To do this, recall the following notation for open/closed intervals:

$$(a,b) = \{t \in \mathbb{R} \mid a < t < b\}$$
$$[a,b] = \{t \in \mathbb{R} \mid a \leq t \leq b\}$$
$$[a,b) = \{t \in \mathbb{R} \mid a \leq t < b\}$$

In particular, we note that $(-\infty, +\infty) = \mathbb{R}$.

Wikipedia: curve

Definition 2.31 (Curve) Let $I \subseteq \mathbb{R}$ be an interval of the real numbers. A *curve* C in $\mathbb{A}^n$ is the image of a continuous map $\boldsymbol{\gamma} : I \to \mathbb{A}^n$,

$$C = \{\mathbf{P} \in \mathbb{A}^n \mid \exists\, t \in I \text{ such that } \mathbf{P} = \boldsymbol{\gamma}(t)\}. \tag{2.14}$$

Wikipedia: parametrisation

The map $\boldsymbol{\gamma}$ is a *parametrisation* for C.

When working in Cartesian coordinates, it will be convenient to write our parametrisation maps as

$$\boldsymbol{\gamma}(t) =$$

where each $\gamma_i : I \to \mathbb{R}$ is a continuous map to the reals.

Example 2.32 Consider the parametrisation map

$$\boldsymbol{\gamma} : [0, 2\pi) \to \mathbb{A}^2$$
$$t \mapsto \tag{2.15}$$

If we were to compute this parametrisation map, we'd see that the image of γ is the circle of the radius a,

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \;\middle|\; x^2 + y^2 = a^2 \right\}. \tag{2.16}$$

Remark 2.33 The following viewpoint is helpful when considering parametrisations of curves. If we consider the parameter t as "time", we can consider a parametrisation as a point moving along a path in space. The curve is the path that is traced out by this point moving.

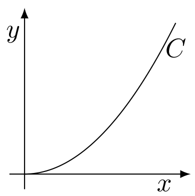
2.4.2 Reparametrisation and orientation

Curves may also be defined as a set of points that satisfy certain conditions or equations. These are sometimes known as *implicit curves*. If we want a parametrisation for these curves, we have to pick it ourselves.

Wikipedia: implicit curves

Example 2.34 Consider the following curve in $\mathbb{A}^2$

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \;\middle|\; y = x^2 \,,\, 0 < x < 1 \right\}.$$



Find a parametrisation for C.

We would like a way to see when two parametrisation maps give rise to the same curve. Furthermore, we would like a controlled way of transforming one parametrisation into another. Both of these can be accomplished via a *reparametrisation map*.

Definition 2.35 (Reparametrisation) Consider the parametrisations $\boldsymbol{\gamma}_1 : I_1 \to \mathbb{A}^n$ and $\boldsymbol{\gamma}_2 : I_2 \to \mathbb{A}^n$. We say that $\boldsymbol{\gamma}_2$ is a *reparametrisation* of $\boldsymbol{\gamma}_1$ if there exists a ____________ bijective map $\varphi \colon I_1 \to I_2$ such that $\forall\ t \in I_1$:

- ____________________________,
- ____________.

We call φ a *reparametrisation map*.

The intuition we should have behind this definition is that φ "deforms" the time interval I_1 into I_2. The first condition states that this deformation makes the two parametrisations equal, implying that they must parametrise the same curve. The second condition states that this deformation cannot "stop time" in one of the intervals, and will have more implications when considering the orientation of curves.

Example 2.36 Recall the two parametrisations $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ from Example 2.34. These give rise to the same curve, and seem relatively well behaved, so we expect $\boldsymbol{\gamma}_2$ to be a reparametrisation of $\boldsymbol{\gamma}_1$. Show that $\boldsymbol{\gamma}_2$ is a reparametrisation of $\boldsymbol{\gamma}_1$.

Example 2.37 We can introduce a third parametrisation of C via the reparametrisation map

$$\psi\colon \left(0, \frac{\pi}{2}\right) \to (0,1)$$
$$t \mapsto$$

We define γ_3 by deforming the parametrisation γ_1 via ψ:

$$\gamma_3\colon \left(0, \frac{\pi}{2}\right) \to \mathbb{A}^2$$
$$t \mapsto \gamma_1(\psi(t)) =$$

Note that γ_3 is clearly a new parametrisation as its domain is a different interval to γ_1 and γ_2.

Remark 2.38 In Example 2.37, as the map ψ goes between I_3 to I_1, the reparametrisation process deforms γ_3 into γ_1. Therefore we say that ____. This may seem counterintuitive, as we had to define γ_3 via γ_1. We are actually doing the following: as we know the behaviour of γ_1 and as ψ is bijective, we can look at its inverse to see what the behaviour of γ_3 must be to deform into γ_1.

If a curve has endpoints $\mathbf{P}, \mathbf{Q}$, a parametrisation can traverse the curve either from $\mathbf{P}$ to $\mathbf{Q}$ or from $\mathbf{Q}$ to $\mathbf{P}$. While these two parametrisations give rise to the same curve, it is helpful to distinguish that they traverse the curve in opposite directions. This gives rise to the notion of the *orientation* of a curve.

Definition 2.39 (Orientation of a curve) Let $\gamma_1\colon I_1 \to \mathbb{A}^n$, $\gamma_2\colon I_2 \to \mathbb{A}^n$ be parametrisations with a reparametrisation map $\varphi\colon I_1 \to I_2$.

- We say γ_1, γ_2 have the *same orientation* if ____________.
- We say γ_1, γ_2 have the *opposite orientation* if ____________.

An *orientation* of the curve C is an equivalence class of parametrisations with the same orientation.

Wikipedia: orientation

As with orientation of bases, the parametrisations of a curve with the same orientation form an equivalence class. Picking a parametrisation for a curve fixes its orientation: if we want to reparametrise and preserve orientation then we must remain in this equivalence class.

Example 2.40 The parametrisations γ_1, γ_2 from Example 2.36 have the same orientation as the reparametrisation map φ has positive derivative for all $t \in (0,1)$. Intuitively this makes sense, as they both begin at the point $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ and end at the point $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$.

Consider another parametrisation $\boldsymbol{\gamma}_4$ of C defined by the reparametrisation map ξ and deforming in $\boldsymbol{\gamma}_1$:

$$\begin{aligned} \xi\colon (0,1) &\to (0,1) & \boldsymbol{\gamma}_4\colon (0,1) &\to \mathbb{A}^2 \\ t &\mapsto 1-t & t &\mapsto \boldsymbol{\gamma}_1(\xi(t)) = \begin{pmatrix} 1-t \\ (1-t)^2 \end{pmatrix} \end{aligned}$$

Does it have the same or opposite orientation as γ_1?

2.4.3 Differential properties of curves

To define the correct notion of a tangent vector and tangent space to a curve, we need to go via *velocity vectors*.

Definition 2.41 (Velocity vector of a parametrisation) Let $\boldsymbol{\gamma}\colon I \to \mathbb{A}^n$ be a parametrisation for a curve C. The *velocity vector of* $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(t_0)$ is

$$\boldsymbol{\gamma}'(t_0) = \qquad\qquad \gamma_i'(t_0) = \tag{2.17}$$

Remark 2.42 The name velocity vector comes from the idea of a parametrisation $\boldsymbol{\gamma}$ being a point moving through space. The velocity vector $\boldsymbol{\gamma}'(t_0)$ is precisely the velocity of the point at time t_0.

Note that we can consider $\boldsymbol{\gamma}'(t)$ as a vector field on the curve that assigns to every point $\boldsymbol{\gamma}(t_0)$ its velocity vector $\boldsymbol{\gamma}'(t_0)$.

Example 2.43 Recall the parametrisation $\boldsymbol{\gamma}$ of the circle with radius a from Example 2.32,

$$\begin{aligned} \boldsymbol{\gamma} : [0, 2\pi) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}. \end{aligned} \tag{2.18}$$

What is the velocity vector of $\boldsymbol{\gamma}$ at time t?

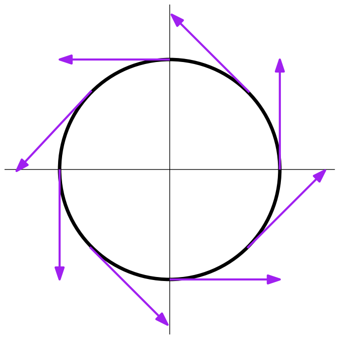


Figure 7: The velocity vectors of the parametrisation from Example 2.43.

A velocity vector is always tangent to the curve at that point. Suppose we reparametrise C, what happens to the velocity vector at a point? The magnitude of the velocity vector may change, but the direction (up to sign) will stay the same.

Definition 2.44 (Tangent vector and space to a curve) A *tangent vector* to C at $\mathbf{P}$ is a velocity vector $\boldsymbol{\gamma}'(t_0)$ such that $\boldsymbol{\gamma}$ is a ______________________ of C and $\mathbf{P} = \boldsymbol{\gamma}(t_0)$.

The *tangent space* to C at $\mathbf{P}$ is the set $T_{\mathbf{P}}(C)$ of tangent vectors to C at $\mathbf{P}$.

Note that $T_{\mathbf{P}}(C) \subset T_{\mathbf{P}}(\mathbb{A}^n)$: the tangent space to a curve is a subspace of the tangent space to $\mathbb{A}^n$. We shall always consider velocity vectors in the tangent space to the curve it parametrises.

Example 2.45 Let C be the circle of radius a and the point $\mathbf{P} = (-a, 0)^{\mathsf{T}}$ on it. Reconsider the parametrisation $\boldsymbol{\gamma}$ of C given in equation (2.18). Is its velocity vector a tangent vector?

We consider a different parametrisation $\widetilde{\boldsymbol{\gamma}}$ of C via the reparametrisation map φ:

$$\varphi\colon \left[0, \frac{2\pi}{k}\right) \to [0, 2\pi) \qquad \widetilde{\boldsymbol{\gamma}}\colon \left[0, \frac{2\pi}{k}\right) \to \mathbb{A}^n$$
$$t \mapsto kt \qquad\qquad t \mapsto \boldsymbol{\gamma}(\varphi(t)) = \begin{pmatrix} a\cos(kt) \\ a\sin(kt) \end{pmatrix}$$

where $k \in \mathbb{R} \setminus 0$. Note if $k < 0$, then $\frac{2\pi}{k} < 0$ and so we rewrite the interval as $\left(\frac{2\pi}{k}, 0\right]$.

Is $\widetilde{\boldsymbol{\gamma}}$ a tangent vector?

We can't realise the zero vector as via the reparametrisation map φ as it is not defined for $k = 0$. However we can still find a parametrisation of C which has zero velocity at $\mathbf{P}$.

While curves are some of the simplest geometric objects, they can still get quite horrible to work with if we are not careful when picking parametrisations.

Definition 2.46 (Smooth parametrisations and curves) Let $\boldsymbol{\gamma}\colon I \to \mathbb{A}^n$ be a parametrisation for a curve. We say $\boldsymbol{\gamma}$ is *smooth* if every $\gamma_i\colon I \to \mathbb{R}$ is smooth, *i.e.*, $\frac{\mathrm{d}^k \gamma_i}{\mathrm{d}t^k}$ is well defined for all positive integers k and for all $t \in I$.

A curve C is smooth if it has a smooth parametrisation.

Example 2.47 Consider the curve C from Example 2.34, but with the endpoints included. Is the parametrisation of C

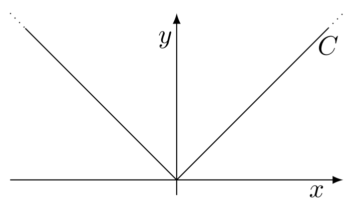
$$\boldsymbol{\gamma}_1\colon [0,1] \to \mathbb{A}^2$$
$$t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

smooth?

Is there (another) parametrisation of C that is not smooth?

Example 2.48 Consider the curve

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; y = |x| \right\}.$$



Is this curve smooth?

Definition 2.49 (Regular parametrisation) A parametrisation $\boldsymbol{\gamma}$ is *regular* if $\boldsymbol{\gamma}'(t)$_______ for all $t \in I$.

Note that all curves we will consider will always have a regular parametrisation.

To define a curve with no regular parametrisation is difficult and all known examples are not natural!

Example 2.50 Consider the parametrisation

$$\begin{aligned} \boldsymbol{\gamma} \colon [0, \pi] &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} \sin t \\ \sin^2 t \end{pmatrix}, \quad \boldsymbol{\gamma}'(t) = \begin{bmatrix} \cos t \\ 2 \sin t \cos t \end{bmatrix}. \end{aligned}$$

This is a parametrisation for the curve C from Example 2.34, but with the endpoints $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ included. Is this parametrisation regular?

Finally, we would like to know how a 0-form f changes as we follow a curve.

Definition 2.51 (Directional derivative along a curve) Let $C \subset \mathbb{A}^n$ be a curve with parametrisation $\boldsymbol{\gamma}$. Let $f : \mathbb{A}^n \to \mathbb{R}$ a 0-form, the *directional derivative of f along $\boldsymbol{\gamma}$* is

Wikipedia: directional derivative of f along $\boldsymbol{\gamma}$

$$D_{\boldsymbol{\gamma}'} f = \tag{2.19}$$

Recall that $\boldsymbol{\gamma}'$ can be considered a vector field on C that assigns to every point its velocity vector. The notation $D_{\boldsymbol{\gamma}'} f$ hints that this should be the directional derivative of f along the vector field $\boldsymbol{\gamma}'$. As $x_i = \gamma_i(t)$, we can use the chain rule to show this is true:

$$D_{\boldsymbol{\gamma}'} f = \tag{2.20}$$

and so $D_{\boldsymbol{\gamma}} f$ is just the directional derivative of f in the direction of the velocity vector.

Remark 2.52 In general, the directional derivative along a curve is sensitive to which parametrisation we pick. If we pick a parametrisation that traverses the curve much "faster", the magnitude of the velocity vectors will be greater, therefore the directional derivative will be greater.