Week 6

2.2 Vector fields

We briefly recall a key object in vector calculus.

Wikipedia: vector field

Definition 2.12 (Vector field) Let $h_1, \ldots, h_n$ be real-valued functions on $\mathbb{A}^n$. A *vector field* $\mathbf{W}$ on $\mathbb{A}^n$ is a function

Affine space

$$\mathbf{W}(\underbrace{x_1, \ldots, x_n}_{P}) = \underbrace{h_1(x_1, \ldots, x_n)\, \mathbf{e}_1 + \cdots + h_n(x_1, \ldots, x_n)\, \mathbf{e}_n}_{\text{vector}} \tag{2.3}$$

that assigns to each point $\mathbf{P} \in \mathbb{A}^n$ a tangent vector $\mathbf{W}(\mathbf{P}) \in T_{\mathbf{P}}(\mathbb{A}^n)$ from its tangent space. $\to (x_1, \ldots, x_n)$

Example 2.13 Let $\mathbf{W}(x, y) =$ $x\mathbf{e}_x + y\mathbf{e}_y$ be a vector field on $\mathbb{A}^2$. The vector field given by $\mathbf{W}$ is in Figure 4. Here $h_1(x, y) =$ x and $h_2(x, y) =$ y.

Wikipedia: gradient of f

Vector fields often arise in mathematics and physics in the following way. Let $f\colon \mathbb{A}^n \to \mathbb{R}$ be a smooth function, the *gradient of f* is the vector field

nabla

$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

Wikipedia: conservative

$\to W = \nabla f$

A ~~vector field~~ W that arises as the gradient of some function f is called a *conservative* vector field. Geometrically, the vector field ∇f points in the direction in which f increases the most.

Example 2.14 Define

$$f(x, y) = \frac{x^2 + y^2}{2}.$$

$W = \nabla f$

then $\frac{\partial f}{\partial x} =$ $\frac{2x}{2} = x$ and $\frac{\partial f}{\partial y} =$ $\frac{2y}{2} = y$. Therefore $\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y =$ $x\mathbf{e}_x + y\mathbf{e}_y$ is precisely the vector field $\mathbf{W}$ defined in Example 2.13. Therefore $\mathbf{W}$ is conservative. Note that the vector field points in the direction that f increases the most, *i.e.*, orthogonally to the level sets.

Wikipedia: level set

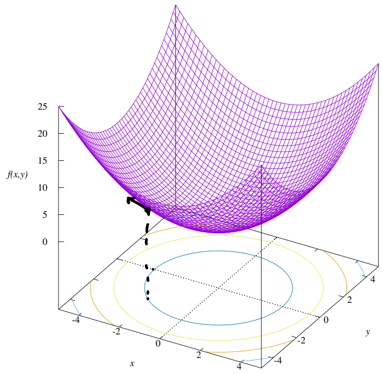


Figure 3: The graph of the function $f = \frac{x^2+y^2}{2}$ with level sets shown below.

Example 2.15 Is the vector field $\widetilde{\mathbf{W}}(x, y) = \underbrace{-y\mathbf{e}_x + x\mathbf{e}_y}$ (shown in Figure 4) is conservative?

If conservative $\exists f$ s.t. $\widetilde{W} = \nabla f$

$\frac{\partial f}{\partial x} = -y \qquad \frac{\partial f}{\partial y} = x$

$-1 = \frac{\partial f}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y} = 1$ Not conservative :(

$\int -y \, dx = -yx + c_x \qquad \int x \, dy = xy + c_y$

$\neq$

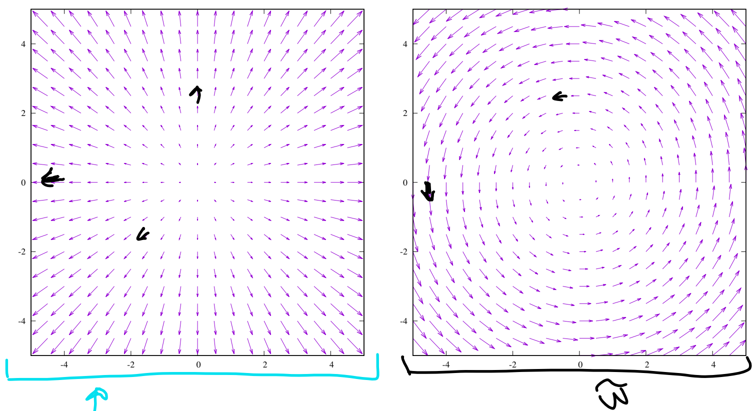


Figure 4: Two vector fields in $\mathbb{A}^2$. The left is the conservative vector field $\mathbf{W} = x\mathbf{e}_x + y\mathbf{e}_y$. The right is the non-conservative vector field $\widetilde{\mathbf{W}} = -y\mathbf{e}_x + x\mathbf{e}_y$.

We briefly recall the notion of the directional derivative of a function.

Definition 2.16 (Directional derivative) Let f be a smooth function on $\mathbb{A}^n$ and $\mathbf{v} \in T_{\mathbf{P}}(\mathbb{A}^n)$ a tangent vector to a point $\mathbf{P}$. The *directional*

Wikipedia: directional derivative of f at $\mathbf{P}$ along $\mathbf{v}$

pt. the direction vector

derivative of f at $\mathbf{P}$ *along* $\mathbf{v}$ is

fn

$$D_{\mathbf{v}}f(\mathbf{P}) = \nabla f(P)\cdot v = \frac{\partial f}{\partial x_1}(P)\, v_1 + \cdots + \frac{\partial f}{\partial x_n}(P)\, v_n \tag{2.4}$$

$W(P) = h_1 \xi_1 + \cdots + h_n \xi_n$, $h_1(x_1, \ldots, x_n)$

More generally, if $\mathbf{W}$ is a vector field of the form (2.3) then the directional derivative of f along the vector field is

$$D_{\mathbf{W}}f = \frac{\partial f}{\partial x_1} h_1 + \cdots + \frac{\partial f}{\partial x_n} h_n \tag{2.5}$$

where evaluating at $\mathbf{P}$ gives you the directional derivative $D_{\mathbf{W}(\mathbf{P})}f(\mathbf{P})$ of f at $\mathbf{P}$ along $\mathbf{W}(\mathbf{P})$.

Example 2.17 Consider again the function $\boxed{f = \frac{x^2+y^2}{2}}$. Its graph is given in Figure 3. For the vector field $\mathbf{W}(x, y) = x\mathbf{e}_x + y\mathbf{e}_y$, what is the directional derivative of f along $\mathbf{W}$?

$$D_W f = \frac{\partial f}{\partial x}\overset{x}{h_x} + \frac{\partial f}{\partial y}\overset{y}{h_y} = x\cdot x + y\cdot y = x^2+y^2$$

$\therefore$ the slope of f increases quadratically as we get further from the origin.

For the vector field $\widetilde{\mathbf{W}}(x, y) = -y\mathbf{e}_x + x\mathbf{e}_y$, what is the directional derivative of f along $\widetilde{\mathbf{W}}$?

$$D_{\widetilde{W}} f = \frac{\partial f}{\partial x} h_x + \frac{\partial f}{\partial y} h_y = x\cdot(-y) + y\cdot x = -xy + xy = 0$$

as we move along $\widetilde{W}$, the value of f doesn't change.

2.3 Differential 0-forms and 1-forms

2.3.1 Definitions

Differential forms are a very powerful, and yet subtle, approach to multivariable calculus and differential geometry. Their purpose is to unify methods of integrating over curves, surfaces and multi-dimensional objects, as well as providing an approach that is independent of choosing coordinates in the space. Unfortunately, their full power is outside the scope of this course and will have to be covered in future courses. Instead, we intend to give an introduction to 0-forms and 1-forms, with an emphasis on how to compute with them, and leave the full gory details for later.

We shall begin by defining 0-forms and 1-forms. We note that the connection between them shall not be immediately apparent, but will be covered in Subsubsection 2.3.2.

Definition 2.18 (0-form) A *differential 0-form* is a smooth real-valued function $f\colon \mathbb{A}^n \to \mathbb{R}$.

all partial derivatives exist.

Wikipedia: differential 0-form

Before introducing differential 1-forms, we consider the following linear maps on the tangent space

$$\mathrm{d}x_i\colon T_{\mathbf{P}}(\mathbb{A}^n) \to \mathbb{R} \tag{2.6}$$

$$\underline{v} = \sum_{i=1}^{n} v_i \underline{e}_i \longmapsto \mathrm{d}x_i(v) = v_i \tag{2.7}$$

$\in T_P(\mathbb{A}^n)$

that maps a vector to its i^{th} entry. We call these *elementary forms*, and these are going to be our building blocks for differential 1-forms.

Wikipedia: elementary forms

Remark 2.19 The classical approach to calculus uses $\mathrm{d}x_i$ to denote an infinitesimal in the x_i coordinate direction. However, the modern viewpoint is that $\mathrm{d}x_i$ should not be an infinitesimal, rather an element of the tangent space in the x_i coordinate direction.

A *linear functional* f on a vector space V is a linear map that maps to $\mathbb{R}$. Note that the elementary forms $\mathrm{d}x_i$ are linear functionals on $T_{\mathbf{P}}(\mathbb{A}^n)$, as are linear combinations of them.

Wikipedia: linear functional

Definition 2.20 (1-form) Let $g_1, \ldots, g_n$ be smooth real-valued functions on $\mathbb{A}^n$. A *differential 1-form* on $\mathbb{A}^n$ is a function

Equivalently, we can assume each g_i is a 0-form.

Wikipedia: differential 1-form

$$\omega = g_1(x_1, \ldots, x_n)\,\mathrm{d}x_1 + \cdots + g_n(x_1, \ldots, x_n)\,\mathrm{d}x_n \tag{2.8}$$

that associates to each point $\mathbf{P} \in \mathbb{A}^n$ a linear functional $\omega(\mathbf{P}, -)$ on its tangent space $T_{\mathbf{P}}(\mathbb{A}^n)$

$$\omega(\mathbf{P}, -)\colon T_{\mathbf{P}}(\mathbb{A}^n) \to \mathbb{R}$$
$$\mathbf{v} \mapsto \omega(\underline{P}, \underline{v}) = g_1(P)\,\mathrm{d}x_1(v) + \cdots + g_n(P)\,\mathrm{d}x_n(v)$$

We shall sometimes denote this linear functional as $\omega_{\mathbf{P}} := \omega(\mathbf{P}, -)$.

$\omega_{\underline{P}}(\underline{v}) = \omega(\underline{P}, \underline{v})$

Example 2.21 Consider the 1-form $\omega = x\mathrm{d}x + y\mathrm{d}y$ on $\mathbb{A}^2$. Then $g_1(\mathbf{P}) = x$ and $g_2(\mathbf{P}) = y$. At the point $\mathbf{P} = (2, 1)^\mathsf{T}$ we get the linear functional

$$\omega_{\mathbf{P}}(\mathbf{v}) = \omega(\mathbf{P}, \mathbf{v}) = 2\,\mathrm{d}x(v) + 1\,\mathrm{d}y(v) = \boxed{2v_x + v_y}$$

for $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \in T_{\mathbf{P}}(\mathbb{A}^2)$.

Remark 2.22 Just by comparing definitions, we see that 1-forms and vector fields look like very similar objects. In fact, there is a correspondence between them via

$$g_1\mathrm{d}x_1 + \cdots + g_n\mathrm{d}x_n \longleftrightarrow h_1\underline{e}_1 + \cdots + h_n\underline{e}_n$$

Vector fields and 1-forms are in fact *dual* objects, similar to the relationship between row and column vectors. Duality is an abstract but important concept that shows up in all area of mathematics.
Wikipedia: Duality in mathematics

While they look very similar, we reiterate that they are different objects:

- A vector field assigns to every point a vector of $T_{\mathbf{P}}(\mathbb{A}^n)$.
- A 1-form assigns to every point a linear functional on $T_{\mathbf{P}}(\mathbb{A}^n)$.

2.3.2 Differentiating 0-forms

Although we do not cover them in this course, we can define 2-forms to be what we get if we differentiate a 1-form. By extension a k-form is what we get if we differentiate a $(k-1)$-form.

What is the relationship between differential 0-forms and 1-forms? The answer is that we can "differentiate" 0-forms via a differential map d to get 1-forms.

$$\boxed{\text{Differential 0-forms}} \xrightarrow{\mathrm{d}} \boxed{\text{Differential 1-forms}}$$

Definition 2.23 (Differential of a 0-form) Let f be a 0-form on $\mathbb{A}^n$. We define the *differential of f* to be the 1-form

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n \tag{2.9}$$

Wikipedia: total derivative

You may have also come across $\mathrm{d}f$ as the *total derivative* of f. From this viewpoint, 1-forms are generalisations of the total derivative, as not all 1-forms need be of the form $\mathrm{d}f$.

Example 2.24 Consider the 0-form on $\mathbb{A}^2$

$$f(x, y) = \frac{x^2 + y^2}{2}.$$

What is the differential of f?

$$df = \underbrace{\frac{\partial f}{\partial x}}_{x}\,dx + \underbrace{\frac{\partial f}{\partial y}}_{y}\,dy = x\,dx + y\,dy$$

1-forms that can be obtained as the differential of a 0-form are particularly nice, especially when we begin integrating them.

Wikipedia: exact

Definition 2.25 (Exact form) A 1-form ω is called *exact* if there exists a 0-form f such that $\omega = df$.

As an example, note that the 1-form ω from Example 2.21, is equal to the 1-form $\mathrm{d}f$ from Example 2.24. As $\omega = \mathrm{d}f$, it is an exact 1-form.

Not all 1-forms are exact forms, just as not all vector fields are conservative.

Proposition 2.26 *Let $\omega = g_1 \mathrm{d}x_1 + \cdots + g_n \mathrm{d}x_n$ be a 1-form on $\mathbb{A}^n$. If ω is exact then*

g_i

$$\boxed{\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}} \quad \forall 1 \leq i, j \leq n. \tag{2.10}$$

Proof. ω is exact if and only if it is of the form $\omega = \mathrm{d}f$ for some 0-form f, *i.e.*, g_i is of the form

$$\boxed{g_i = \frac{\partial f}{\partial x_i}}.$$

If we differentiate g_i with respect to some x_j, we see that

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial g_j}{\partial x_i}.$$

$i \leftrightarrow j$

□

This proposition is useful for trying to rule out whether a 1-form is exact: if condition (2.10) does not hold then it cannot be exact.

Example 2.27 Consider the 1-form $\omega = \lambda y \mathrm{d}x + x \mathrm{d}y$. When is ω exact? When is it not exact?

$\frac{\partial f}{\partial x}$

- $\frac{\partial}{\partial y}(\lambda y) = \lambda$
- $\frac{\partial (x)}{\partial x} = 1$

Not exact if $\lambda \neq 1$,

Is ω exact when $\lambda = 1$?

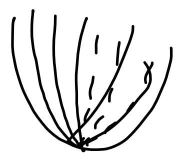
$\int y\,dx = xy + C_x$

$\int x\,dy = xy + C_y$

$\boxed{f = xy} \rightarrow \omega = df$

Remark 2.28 Recall that the purpose of differentiating a function is to give a linear approximation of it at a point. With exact forms, we can now shed some light on the geometric intuition behind 1-forms.

Consider again the 0-form $f = \frac{x^2+y^2}{2}$ from Example 2.24. Its differential $\mathrm{d}f = x\mathrm{d}x + y\mathrm{d}y$ assigns to the point $\mathbf{P} = (1,1)^\mathsf{T}$ the linear functional



$$\mathrm{d}f(\mathbf{P}, -) \colon T_{\mathbf{P}}(\mathbb{A}^n) \to \mathbb{R}$$
$$\mathbf{v} \mapsto v_x + v_y$$

on the tangent space at $\mathbf{P}$. Figure 5 shows the level sets of this linear functional on $T_{\mathbf{P}}(\mathbb{A}^n)$. However, we can also consider the level sets of f locally around $\mathbf{P}$. We see that by comparing, $\underbrace{\mathrm{d}f}_{\omega}(\mathbf{P}, -)$ gives a linear approximation of f at $\mathbf{P}$. This is essentially the purpose of differentiation.

With this intuition, we can think of an exact 1-form $\mathrm{d}f$ as encoding the linear approximation of f at every point $\mathbf{P}$. For a 1-form ω is not exact, it is also encoding many "linear approximations" but there is no 0-form f that can approximated in this way.

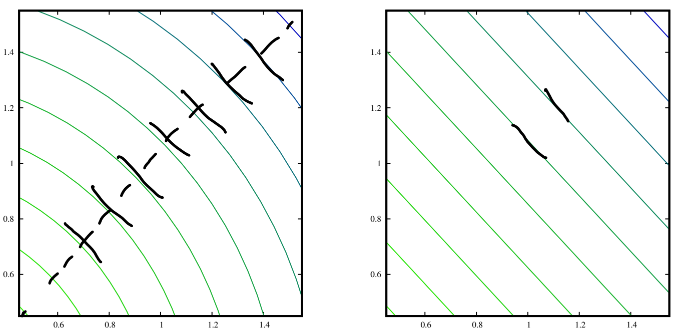


Figure 5: The level sets of the function $f = \frac{x^2+y^2}{2}$ in $\mathbb{A}^2$ on the left, and the level sets of the linear functional $v_x + v_y$ in the tangent space $T_{\mathbf{P}}\left(\mathbb{A}^2\right)$ for $\mathbf{P} = (1,1)^\intercal$ on the right.

2.3.3 Applying 1-forms to vector fields

We saw in the definition of 1-forms that if we fix a point $\mathbf{P}$ and treat the tangent vectors as variables, then $\omega(\mathbf{P}, -)$ is a linear functional. What happens when we fix the tangent vectors and vary the points?

To make this precise, consider a 1-form ω and a vector field $\mathbf{W}$ defined by

$$\omega = g_1(x_1, \ldots, x_n)\mathrm{d}x_1 + \cdots + g_n(x_1, \ldots, x_n)\mathrm{d}x_n,$$
$$\mathbf{W} = h_1(x_1, \ldots, x_n)\mathbf{e}_1 + \cdots + h_n(x_1, \ldots, x_n)\mathbf{e}_n.$$

Ln $dx_1(W) = h_1(x_1,\ldots,x_n)$

The vector field fixes a tangent vector $\mathbf{W}(\mathbf{P})$ for each point $\mathbf{P}$. We can "apply" ω to the vector field $\mathbf{W}$ to define a function on $\mathbb{A}^n$

$$\omega(-, \mathbf{W})\colon \mathbb{A}^n \to \mathbb{R} \tag{2.11}$$
$$\mathbf{P} \mapsto \omega(P, W(P)) \tag{2.12}$$

Using the definition of a 1-form, we can simplify the expression for $\omega(-, \mathbf{W})$ down to

$$\begin{aligned}
\omega(-, \mathbf{W}) &= g_1(x_1, \ldots, x_n)\mathrm{d}x_1(\mathbf{W}) + \cdots + g_n(x_1, \ldots, x_n)\mathrm{d}x_n(\mathbf{W}) \\
&= g_1(x_1, \ldots, x_n)h_1(x_1, \ldots, x_n) + \cdots \\
&\quad + g_n(x_1, \ldots, x_n)h_n(x_1, \ldots, x_n),
\end{aligned} \tag{2.13}$$

$= g_1(P)\, h_1(P) + \cdots + g_n(P)\, h_n(P)$

$\omega(P, W(P)) = \sum_{i=1}^{n} g_i(P)\, h_i(P)$ from 1-form ω W vector

as $\mathrm{d}x_i(\mathbf{W}) = h_i(x_1, \ldots, x_n)$.

Example 2.29 Consider the 1-form $\omega = x\,dx + y\,dy$ and the vector field $\mathbf{W} = -y e_x + x e_y$. Applying equation (2.13), we see that

$$\omega(-, \mathbf{W}) = x \cdot (-y) + y \cdot x = 0.$$

This seems slightly anticlimactic, but we will show there is some nice geometry going on in the background here.

Note that something particularly nice happens when ω is exact. If $\omega = \mathrm{d}f$, then evaluating $\mathrm{d}f(-, \mathbf{W})$ at $\mathbf{P}$ gives

$$\overset{\omega}{\mathrm{d}f(-, \mathbf{W})} = \overset{g_1}{\boxed{\frac{\partial f}{\partial x_1}}} h_1 + \cdots + \overset{g_n}{\boxed{\frac{\partial f}{\partial x_n}}} h_n = \boxed{D_{\mathbf{W}} f},$$

the directional derivative of f along the vector field $\mathbf{W}$. From this point of view, 1-forms are generalisations of directional derivatives of a function along a vector field.

Example 2.30 Let us reconsider Example 2.29. Recall that $\omega = \mathrm{d}f$ where $f = \frac{x^2+y^2}{2}$, a function whose level sets are circles in the plane. The graph of f and the vector field $\mathbf{W}$ are shown again in Figure 4.

As ω is exact, the geometric intuition is that $\omega(-, \mathbf{W})$ is the directional derivative of f along $\mathbf{W}$. Note that a point moving around the vector field $\mathbf{W}$ traces out a circle, and on these circles the value of f does not change. Therefore the directional derivative of f along $\mathbf{W}$ is 0, lining up with our value of $\omega(-, \mathbf{W})$.

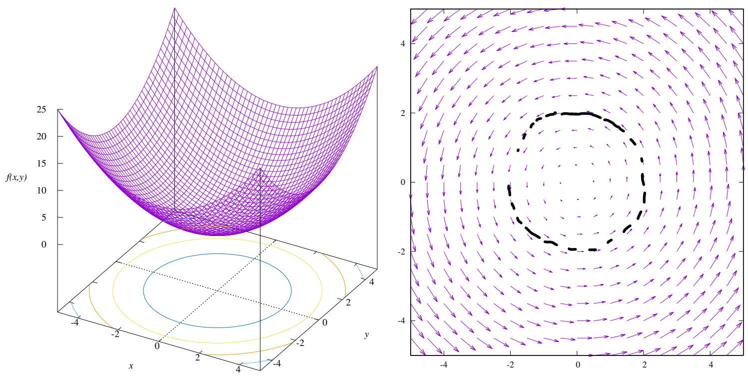


Figure 6: Recalling the 0-form $f = \frac{x^2+y^2}{2}$ and the vector field $\mathbf{W} = -y\mathbf{e}_x + x\mathbf{e}_y$.