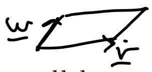
Week 5

1.7.3 Area of a parallelogram

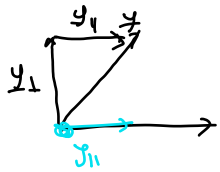
Axiom (VP-Len) states that the length of the vector product of two perpendicular vectors is given by the area of a rectangle bordered by the pair. The following proposition generalises this to the area of the parallelogram formed by two arbitrary vectors.

We shall represent the parallelogram formed by two vectors, $\mathbf{v}$ and $\mathbf{w}$, with the notation $\varDelta(\mathbf{v}, \mathbf{w})$.



Proposition 1.87 *The area of the parallelogram formed by the vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *is given by the length of their vector product,* $\|\mathbf{x} \times \mathbf{y}\|$.

$$\text{Area}(\varDelta(\underline{x}, \underline{y})) = \| \underline{x} \times \underline{y} \| \tag{1.35}$$



Proof. Consider the expansion $\mathbf{y} = \mathbf{y}_\parallel + \mathbf{y}_\perp$, where the vector $\mathbf{y}_\perp$ is orthogonal to the vector $\mathbf{x}$ and the vector $\mathbf{y}_\parallel$ is parallel to vector $\mathbf{x}$. The area of $\varDelta(\mathbf{x}, \mathbf{y})$ is equal to the product of the length of the vector $\mathbf{x}$ (the base) and the length of vector $\mathbf{y}_\perp$ (the height).

On the other $\mathbf{x} \times \mathbf{y} = \mathbf{x} \times (\mathbf{y}_\parallel + \mathbf{y}_\perp) = \mathbf{x} \times \mathbf{y}_\parallel + \mathbf{x} \cancel{\times} \mathbf{y}_\perp$. But $\boxed{\mathbf{x} \times \mathbf{y}_\parallel = 0}$ because these vectors are collinear. Hence $\mathbf{x} \times \mathbf{y} = \mathbf{x} \times \mathbf{y}_\perp = \|\mathbf{x}\|\|\mathbf{y}_\perp\|$ because vectors $\mathbf{x}$ and $\mathbf{y}_\perp$ are perpendicular. $\square$

This proposition is very important in understanding the meaning of the vector product. Succinctly, the vector product of two vectors is a vector that is orthogonal to the plane spanned by these vectors, with magnitude equal to the area of the parallelogram formed by the vectors. The direction of the vector is defined by orientation.

It is worth recalling a formula relating the area of a parallelogram to the length of its sides and the angle between them:

$$\underset{\text{Area of parallelogram}}{\|\mathbf{x} \times \mathbf{y}\|} = \|\underline{x}\|\|\underline{y}\||\sin\theta| \tag{1.36}$$

Compare this to the formula we saw for inner products:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta. \tag{1.7}$$

These two formulas demonstrate a fundamental property of the two different products:

- The inner product is *zero* if the pair of vectors are orthogonal.
- The vector product is *zero* if the pair of vectors are colinear.

In fact equation (1.36) can be derived from equation (1.7) using the identity $\|\mathbf{v} \times \mathbf{w}\|^2 + \langle \mathbf{v}, \mathbf{w} \rangle^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$, which we prove in Lemma 1.88 below. Using this identity we have ?

$$\begin{aligned} \|\mathbf{x} \times \mathbf{y}\|^2 + \langle \mathbf{x}, \mathbf{y} \rangle^2 &= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \\ &= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 (\sin^2 \theta + \cos^2 \theta) \\ &= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \sin^2 \theta + \underbrace{\langle \mathbf{x}, \mathbf{y} \rangle^2}_{\|x\|^2\|y\|^2\cos^2\theta}. \end{aligned}$$

Eliminating the inner product from both sides and taking square roots gives equation (1.36). Thus if we abstractly define angles using the inner product formula, this is consistent with doing so with the vector product formula.

Lemma 1.88 *For a pair of vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{E}^3$ *the following identity holds:*

$$\|\underline{v} \times \underline{w}\|^2 + \langle \underline{v}, \underline{w} \rangle^2 = \|\underline{v}\|^2 \|\underline{w}\|^2$$

Proof. Fix an orthonormal basis $\mathcal{B}$ and let

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

We can be a little lazy with signs in the proof since each determinant is squared.

$$\underline{v} \times \underline{w} = \det \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Using the determinant formula we have the following:

$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= (\det [\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}])^2 + (\det [\begin{smallmatrix} v_1 & v_3 \\ w_1 & w_3 \end{smallmatrix}])^2 + (\det [\begin{smallmatrix} v_2 & v_3 \\ w_2 & w_3 \end{smallmatrix}])^2 \\ &= (v_1 w_2 - v_2 w_1)^2 + (v_1 w_3 - v_3 w_1)^2 + (v_2 w_3 - v_3 w_2)^2 \\ &= (v_1 w_2)^2 + (v_1 w_3)^2 + (v_2 w_1)^2 \\ &\quad + (v_2 w_3)^2 + (v_3 w_1)^2 + (v_3 w_2)^2 \quad \} \, 1 \\ &\quad - 2 v_1 w_1 v_2 w_2 - 2 v_1 w_1 v_3 w_3 - 2 v_2 w_2 v_3 w_3 \quad \} -2 \end{aligned}$$

Calculating the square of the inner product we have:

$$\begin{aligned} \boxed{\langle \mathbf{v}, \mathbf{w} \rangle^2} &= (v_1 w_1 + v_2 w_2 + v_3 w_3)^2 \\ &= (v_1 w_1)^2 + (v_2 w_2)^2 + (v_3 w_3)^2 \quad \} \, 3 \\ &\quad + 2 v_1 w_1 v_2 w_2 + 2 v_1 w_1 v_3 w_3 + 2 v_2 w_2 v_3 w_3 \quad \} \, 2 \end{aligned}$$

Finally the square product of norms gives:

$$\begin{aligned}\|\mathbf{v}\|^2\|\mathbf{w}\|^2 &= (v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2)\\ &= (v_1w_1)^2+(v_2w_2)^2+(v_3w_3)^2 \quad 3\\ &\quad + (v_1w_2)^2+(v_1w_3)^2+(v_2w_1)^2 \quad 1\\ &\quad + (v_2w_3)^2+(v_3w_1)^2+(v_3w_2)^2\end{aligned}$$

We can now see that $\|\mathbf{v}\times\mathbf{w}\|^2+\langle\mathbf{v},\,\mathbf{w}\rangle^2=\|\mathbf{v}\|^2\|\mathbf{w}\|^2$. ☐

$1-2 + 2+3 = 1 + 3$, σ-parts.

1.7.4 Area and determinants in $\mathbb{E}^2$

Let $\mathbf{a}$ and $\mathbf{b}$ be two linearly independent vectors in a 2-dimensional Euclidean vector space, $\mathbb{E}^2$. We can consider the 2-dimensional space as a plane in an oriented 3-dimensional Euclidean space, $\mathbb{E}^3$. Our aim is to calculate the area of the parallelogram $\varnothing(\mathbf{a},\mathbf{b})$ formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

Let $\mathbf{n}$ be a unit vector in $\mathbb{E}^3$ which is orthogonal to $\mathbb{E}^2$, chosen so that the basis $(\mathbf{a},\mathbf{b},\mathbf{n})$ has the same orientation as $\mathbb{E}^3$. Axiom (VP-⊥) means that the vector product in $\mathbf{a}\times\mathbf{b}$ is proportional to the normal vector $\mathbf{n}$:

If we had selected the other unit normal vector $-\mathbf{n}$, the vector product would still have been proportional to $\mathbf{n}$, however we would need to replace α with $-\alpha$.

$$\mathbf{a}\times\mathbf{b} = \alpha\,\mathbf{n}, \qquad \text{where } \alpha \text{ is the area of } \varnothing(\mathbf{a},\mathbf{b}).$$

Let $(\mathbf{e},\mathbf{f})$ be an orthonormal basis for the plane $\mathbb{E}^2$, again chosen in the order so that the orthonormal basis $(\mathbf{e},\mathbf{f},\mathbf{n})$ has the same orientation as $\mathbb{E}^3$. This is equivalently to choosing $(\mathbf{e},\mathbf{f})$ to have the same orientation as $(\mathbf{a},\mathbf{b})$. Let $\mathbf{a}=a_1\mathbf{e}+a_2\mathbf{f}$ and $\mathbf{b}=b_1\mathbf{e}+b_2\mathbf{f}$. Then

$$\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix}\mathbf{e} & \mathbf{f} & \mathbf{n}\\ a_1 & a_2 & 0\\ b_1 & b_2 & 0\end{bmatrix} = \mathbf{n}\,\underbrace{\det\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}}\ \text{area !!} \tag{1.37}$$

Thus $\alpha=\det\left[\begin{smallmatrix}a_1 & a_2\\ b_1 & b_2\end{smallmatrix}\right]$. If we had instead selected a basis with the opposite orientation, for example $(\mathbf{f},\mathbf{e},\mathbf{n})$, we would instead have α equal to the negative of the determinant. Thus if $(\mathbf{e}_1,\mathbf{e}_2)$ is any orthonormal basis for a 2-dimensional Euclidean space, with $\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2$ and $\mathbf{w}=w_1\mathbf{e}_1+w_2\mathbf{e}_2$ arbitrary vectors then

$$\operatorname{Area}(\varnothing(\mathbf{v},\mathbf{w})) = \left|\det\begin{bmatrix}v_1 & v_2\\ w_1 & w_2\end{bmatrix}\right|. \tag{1.38}$$

Next we want to consider the action of a linear operator on the parallelogram formed by two vectors. We shall see in the next proposition that for a linear operator acting on a 2-dimensional space, the determinant of the linear operator controls how the area scales.

Proposition 1.89 *Let $P\colon \mathbb{E}^2 \to \mathbb{E}^2$ be a linear operator, and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in* $\mathbb{E}^2$. *Denote the images under* P *by* $\underbrace{\mathbf{a}' = P(\mathbf{a})}$ *and* $\underbrace{\mathbf{b}' = P(\mathbf{b})}$*. Then*

$$\text{Area}\big(\square(\mathbf{a}', \mathbf{b}')\big) = |\det P| \text{ Area}\big(\square(\underline{a}, \underline{b})\big)$$

Proof. Fix a basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ for $\mathbb{E}^2$ and let

$$[\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad [P]_{\mathcal{B}} = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}.$$

We obtain the coefficients of $\mathbf{a}'$ and $\mathbf{b}'$ by multiplying the matrix by each column vector. This together with equation (1.38) of the last section gives:

after acting w/ P

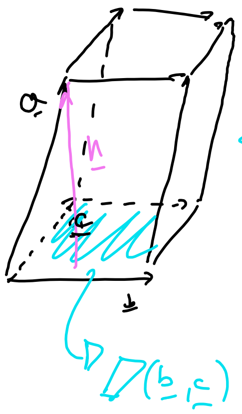
$P(a) = a'$

$P(a_1 \underline{e}_1) = p_{1,1} a_1 \underline{e}_1$

$$\begin{aligned}
\text{Area}\big(\square(\mathbf{a}', \mathbf{b}')\big) &= \left|\det \begin{bmatrix} p_{1,1}a_1 + p_{1,2}a_2 & p_{2,1}a_1 + p_{2,2}a_2 \\ p_{1,1}b_1 + p_{1,2}b_2 & p_{2,1}b_1 + p_{2,2}b_2 \end{bmatrix}\right| \\
&= \left|\det \left(\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \underbrace{\begin{bmatrix} p_{1,1} & p_{2,1} \\ p_{1,2} & p_{2,2} \end{bmatrix}} \right)\right| \\
&= \left|\det P^{\mathsf{T}}\right| \cdot \left|\det \underbrace{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}}\right| \\
&= |\det P| \text{ Area}\big(\square(\mathbf{a}, \mathbf{b})\big)
\end{aligned}$$

∎

1.7.5 Volume and determinants in $\mathbb{E}^3$



The vector product of a pair of vectors is related with area of the parallelogram they form. We will now consider the parallelepiped formed by three vectors.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in $\mathbb{E}^3$. We shall denote the parallelepiped formed by these three vectors with the notation $\square\!\!\square(\mathbf{a}, \mathbf{b}, \mathbf{c})$. We may consider the parallelogram $\square(\mathbf{b}, \mathbf{c})$ as the base of the parallelepiped. The height vector $\mathbf{h}$, is now proportional to $\mathbf{b} \times \mathbf{c}$ (as it is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$) and forms some angle θ, with $\mathbf{a}$.

The volume of $\square\!\!\square(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is equal to the length of the height vector $\mathbf{h}$, multiplied by the area of the base, $\square(\mathbf{b}, \mathbf{c})$.

$$\begin{aligned}
\text{Vol}\big(\square\!\!\square(\underline{a}, \underline{b}, \underline{c})\big) &= \text{Area}\big(\square(\underline{b}, \underline{c})\big)\|\underline{h}\| \\
&= \text{Area}\big(\square(\underline{b}, \underline{c})\big)\, \|\underline{a}\|\, |\cos\theta| \quad \to \text{prop}^{\text{n}}\ 1.87 \\
&= \|\underline{b} \times \underline{c}\|\, \|\underline{a}\|\, |\cos\theta| \\
&= |\langle \underline{a}, \underline{b} \times \underline{c} \rangle|
\end{aligned}$$

Let us express the vectors in terms of an orthonormal basis

$\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in the usual way:

$$[\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad [\mathbf{b}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad [\mathbf{c}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

We can expand the inner and vector products using the chosen basis:

$$\mathrm{Vol}\big(\mathcal{P}(\mathbf{a}, \mathbf{b}, \mathbf{c})\big) = |\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle|$$

$$= \left| \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} \det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} \\ -\det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} \\ \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \end{bmatrix} \right|$$

$$= \left| a_1 \det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - a_2 \det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + a_3 \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \right|$$

$$= \left| \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

Putting these together we come to the beautiful formula:

$$\mathrm{Vol}\big(\mathcal{P}(\mathbf{a}, \mathbf{b}, \mathbf{c})\big) = |\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle| = \left| \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right|. \qquad (1.39)$$

Remark 1.90 Just as we remarked for the area of a parallelogram, sometimes it is useful to consider the *algebraic area* of a parallelepiped as either positive or negative. In this situation, we would define the volume to be $\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle$ without taking the absolute value. The sign would then depend on the orientation of the vector space.

We can now state and prove a proposition for linear operators and volumes in $\mathbb{E}^3$, analogous to Proposition 1.89 that considered areas and operators in $\mathbb{E}^2$.

Proposition 1.91 *Let $P\colon \mathbb{E}^3 \to \mathbb{E}^3$ be a linear operators and let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{E}^3$. Denote the images under P by $\mathbf{a}' = P(\mathbf{a})$, $\mathbf{b}' = P(\mathbf{b})$ and $\mathbf{c}' = P(\mathbf{c})$. Then*

$$\mathrm{Vol}\big(\mathcal{P}(\mathbf{a}', \mathbf{b}', \mathbf{c}')\big) = |\det P| \, \mathrm{Vol}\big(\mathcal{P}(\mathbf{a}, \mathbf{b}, \mathbf{c})\big) \qquad (1.40)$$

Proof. The arguments in the proof of Proposition 1.89 can be applied, mutatis mutandis, to the three dimensional case.

More succinctly (using $\det M = \det M^{\mathsf{T}}$) we can see that, having fixed an orthonormal basis $\mathcal{B}$, the volume is given by

$$\begin{aligned}
\text{Vol}\big(\text{▱}(\mathbf{a}', \mathbf{b}', \mathbf{c}')\big) &= \left|\det\begin{bmatrix}[\mathbf{a}']_{\mathcal{B}} & [\mathbf{b}']_{\mathcal{B}} & [\mathbf{c}']_{\mathcal{B}}\end{bmatrix}\right| \\
&= \left|\det\Big([P]_{\mathcal{B}}\underbrace{\begin{bmatrix}[\mathbf{a}]_{\mathcal{B}} & [\mathbf{b}]_{\mathcal{B}} & [\mathbf{c}]_{\mathcal{B}}\end{bmatrix}}\Big)\right| \\
&= |\det P|\,\text{Vol}\big(\text{▱}(\mathbf{a}, \mathbf{b}, \mathbf{c})\big).
\end{aligned}$$

□

2 Differential geometry

2.1 Affine space

Section 1 was entirely focused on vectors and vector spaces. However, geometry often deals with spaces whose the elements are "points". Moreover, we would like a space where both points and vectors can interact with one another. This leads to the notion of *affine spaces*.

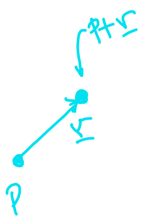
Wikipedia: Euclidean affine space

Definition 2.1 (Euclidean affine space) Let $\mathbb{E}^n$ be an n-dimensional Euclidean vector space. A *Euclidean affine space* (associated with $\mathbb{E}^n$) is a set of points $\mathbb{A}^n$, along with an addition map that allows us to add points with vectors

$\underline{v} \in \mathbb{E}^n$

$$\begin{aligned}
\mathbb{A}^n \times \mathbb{E}^n &\to \mathbb{A}^n \\
(\mathbf{P}, \mathbf{v}) &\mapsto \boxed{\mathbf{P} + \mathbf{v}}
\end{aligned}$$

point in $\mathbb{A}^n$



such that

(1) $\forall\, \mathbf{v}, \mathbf{w} \in \mathbb{E}^n, \mathbf{P} \in \mathbb{A}^n$, $P + (\underline{v} + \underline{w}) = (P + \underline{v}) + \underline{w}$

(2) $\forall\, \mathbf{P} \in \mathbb{A}^n$, $P + \underline{0} = P$

(3) $\forall\, \mathbf{P}, \mathbf{Q} \in \mathbb{A}^n$, $\exists$ a unique $\mathbf{v} \in \mathbb{E}^n$ such that $P + \underline{v} = Q$

Note that as they behave differently, we denote points in uppercase and vectors in lower case.

This definition may seem a bit abstract and unintuitive, but in fact $\mathbb{A}^n$ behaves exactly how we expect $\mathbb{R}^n$ to behave when doing geometry. We can add a point and a vector to get to a new point, we can add two vectors to get a new vector, but we cannot add points together.

Remark 2.2 Unlike vector spaces, affine spaces do not come with a fixed "zero" or "origin". If we want to use one, we have to make this choice.

Let $\mathbb{A}^n$ be a Euclidean affine space with associated vector space $\mathbb{E}^n$ with orthonormal basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$. We wish to find a way to describe the points of $\mathbb{A}^n$. This leads to the notion of *a coordinate system on* $\mathbb{A}^n$.

$n = 3$

$(1, 3, 2.7) \longrightarrow P \in \mathbb{A}^3$

we can think of P as "$(1,3,2.7)$"

Definition 2.3 (Coordinate system) A *coordinate system on* $\mathbb{A}^n$ is a surjective map $\mathbb{R}^n \to \mathbb{A}^n$ that assigns every n-tuple of real numbers to a point $\mathbf{P}$ in $\mathbb{A}^n$.

Wikipedia: Coordinate system
Wikipedia: Coordinates in affine space

In the definition of a coordinate system, $\mathbb{R}^n$ is used without any vector space structure: every element is just a list of n real numbers.

Note that there are many different choices of coordinate system, but we shall begin with the most natural choice, *Cartesian coordinates.*

Example 2.4 (Cartesian coordinates) We first pick some arbitrary point $\mathbf{O} \in \mathbb{A}^n$ to act as an "origin" of the space, as well as an orthonormal basis $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ for $\mathbb{E}^n$. The *Cartesian coordinate system* assigns the n-tuple of real numbers $(x_1, \dots, x_n)$ to the point

Wikipedia: Cartesian coordinate system

$$\mathbf{P} = O + \underline{v} = O + x_1 \underline{e}_1 + \dots + x_n \underline{e}_n \tag{2.1}$$

where $\mathbf{v} = \sum_i x_i \mathbf{e}_i$ is some vector in $\mathbb{E}^n$. Note that by (3), there always exists some vector $\mathbf{v}$ such that $\mathbf{P} = \mathbf{O} + \mathbf{v}$, and so every point has some n-tuple associated with it.

Once $\mathbf{O}$ and $\mathcal{B}$ have been fixed, the Cartesian coordinate representation of $\mathbf{P}$ is the column vector

consider

$$\underline{\mathbf{P}} \text{``}=\text{''} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\text{point}}. \tag{2.2}$$

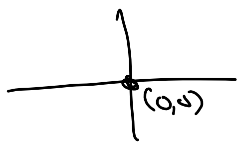
Points = parenthesis

Points will have lines under them from now on

$$\overset{\text{lin op.}}{[P]_{\mathcal{B}}} = \underbrace{\begin{bmatrix} P_{ij} \end{bmatrix}}_{\text{vectors}}$$

Unless stated otherwise, we shall work with Cartesian coordinates. Note that we will use round brackets to denote that (2.2) represents a point rather than a vector.

Remark 2.5 Example 2.4 leads to exactly the same notion of Cartesian coordinates we are familiar with on $\mathbb{R}^n$: we have a coordinate axes spanned by $\mathcal{B}$ centered at the origin $\mathbf{O}$. The only difference is we had to explicitly choose $\mathbf{O}$ and $\mathcal{B}$.



Remark 2.6 A warning about literature: many authors use $\mathbb{R}^n$ for both the affine space and corresponding vector space. However this can lead to confusion, and so we shall stick with $\mathbb{A}^n$ to emphasise the difference with the vector space $\mathbb{E}^n$.

Example 2.7 (Polar coordinates) We know that Cartesian coordinates are not the only choice of coordinates in $\mathbb{A}^n$. For example, we can define *polar coordinates* on $\mathbb{A}^2$ in the following way. Fix an origin $\mathbf{O} \in \mathbb{A}^2$ and an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, the tuple (r, θ) gets mapped to the point $\mathbf{P} = \mathbf{O} + \mathbf{v}$ where $\mathbf{v}$ has magnitude r and angle θ with $\mathbf{e}_1$, *i.e.*,

Wikipedia: polar coordinates

$$r = \|\underline{v}\|, \quad \langle \mathbf{v}, \mathbf{e}_1 \rangle = \|v\| \cos\theta.$$

However we can also describe polar coordinate by how it relates to Cartesian coordinates. In particular, if (x, y) is the Cartesian representation,

$(x, y) \Rightarrow \underline{v} = x\underline{e}_1 + y\underline{e}_2$

$\downarrow$

$(r, \theta) \Rightarrow \|\underline{v}\| = r$

$\langle \underline{v}, \underline{e}_1 \rangle = r\cos\theta$

it is related to polar coordinates via

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

There are many more coordinate systems we may pick, however the underlying geometry should stay the same regardless of our choice. Therefore we shall work with Cartesian coordinates and show our methods hold for arbitrary choices of coordinates later.

Note Unless stated otherwise, we shall fix an origin $\mathbf{O} \in \mathbb{A}^n$ and an orthonormal basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ for $\mathbb{E}^n$, and use Cartesian coordinates on $\mathbb{A}^n$ with respect to $\mathbf{O}, \mathcal{B}$.

2.1.1 Tangent spaces

In vector spaces, all vectors began at the same point. This is not the case with affine space, we have to specify which point a vector is based at. This leads to the notion of tangent vectors and tangent spaces.

Wikipedia: tangent vector

Wikipedia: tangent space of $\mathbb{A}^n$ at $\mathbf{P}$

Definition 2.8 (Tangent vectors and tangent spaces) Let $\mathbf{P}$ be a point in $\mathbb{A}^n$. A *tangent vector* $\mathbf{v_P}$ to $\mathbb{A}^n$ is a vector $\mathbf{v} \in \mathbb{E}^n$ beginning at the point $\mathbf{P} \in \mathbb{A}^n$.

The *tangent space of* $\mathbb{A}^n$ *at* $\mathbf{P}$ is the set $T_{\mathbf{P}}(\mathbb{A}^n)$ of all tangent vectors $\mathbf{v_P}$ to $\mathbb{A}^n$ beginning at $\mathbf{P}$.

We first note that the tangent space $T_{\mathbf{P}}(\mathbb{A}^n)$ is a vector space, as adding and scaling vectors does not change their base point. Furthermore, as we can add any vector in $\mathbb{E}^n$ to a point $\mathbf{P}$, the vector space $T_{\mathbf{P}}(\mathbb{A}^n)$ is a copy of $\mathbb{E}^n$, *i.e.*, $T_P(\mathbb{A}^n) \cong \mathbb{E}^n$.

As $T_{\mathbf{P}}(\mathbb{A}^n)$ is a vector space, we can describe it with a basis. In theory, we may pick a different basis $\mathcal{B}_{\mathbf{P}}$ for each tangent space $T_{\mathbf{P}}(\mathbb{A}^n)$, and we shall see later that for some coordinate systems this is the correct thing to do. However, as we are working with Cartesian coordinates on $\mathbb{A}^n$, **we shall fix the same basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ for every tangent space $T_{\mathbf{P}}(\mathbb{A}^n)$.** This allows us to write elements of $T_{\mathbf{P}}(\mathbb{A}^n)$ as column vectors (with square brackets) as we did in $\mathbb{E}^n$.

$\mathcal{B} = \mathcal{B}_P$
$\forall P \in \mathbb{A}^n$

Remark 2.9 The name tangent vector and tangent space may seem odd here, considering they don't appear to be "tangential" to anything. This connection will become more apparent when we define tangent vectors to curves and surfaces.

Remark 2.10 When doing vector calculus in $\mathbb{R}^3$, you may have come across the notation where tangent vectors are given by $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. This gives a vector of the tangent space: without knowing the point at which the vector begins, it does not make sense on it's own.

Finally, we make a quick note about functions on affine space. As we are doing differential geometry, we want our functions to be as differentiable as possible!

Definition 2.11 (Smooth functions) Let f be a real-valued function on $\mathbb{A}^n$

$$f\colon \mathbb{A}^n \to \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto f(x_1, \ldots, x_n).$$

We say f is *smooth* if every partial derivative of f exists, *i.e.*,

Wikipedia: smooth

$$\frac{\partial^a f}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}}$$ exists for all $a_i \in \mathbb{Z}_{\geq 0}$, $a_1 + \cdots + a_n = a$.

We won't need to worry too much about this definition: all this means for us is we are free to differentiate smooth functions as much as we want and not worry about awkward issues such as whether a derivative exists or not. For us, practically all the functions we will work with will be smooth.