Week 3

1.5 Orientation of vector spaces

You may have heard the term *orientation* before. In particular, you may have heard phrases such as:

> The basis $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ has the same orientation as the basis $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ if they both obey right hand rule or if they both obey left hand rule.

or

> A mirroring image has the opposite orientation to its source.

In this section we try to give exact meaning to these ideas.

Definition 1.59 (Same/opposite orientation) Let $\mathcal{B}$ and $\mathcal{C}$ be two bases for a vector space V and let T be the transition matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$.

↳ ${}_{\mathcal{B}}T_{\mathcal{E}}$

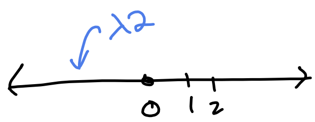
Wikipedia: Orientation

We say that $\mathcal{C}$ has *the same orientation* as $\mathcal{B}$ if $\det T > 0$.
We say that $\mathcal{C}$ has *an opposite orientation* to $\mathcal{B}$ if $\boxed{\det T < 0}$.

Recall that a transition matrix between bases is nondegenerate, hence its determinant cannot be equal to zero.

is a line

Example 1.60 The simplest example is that of a line, $\mathbb{E}$, a 1-dimensional vector space. Any non-zero element of $\mathbb{E}$ spans the space, so let us consider the following three bases, each with a single element:



$$\underbrace{\mathcal{B} = (2)}_{e_1}, \quad \underbrace{\mathcal{C} = (-8)}_{f_1}, \quad \underbrace{\mathcal{D} = (10)}$$

Which basis have the same orientations and which have opposite orientation?

$${}_{\mathcal{B}}T_{\mathcal{E}} = [-4]$$

$$-8 = (-4)2$$
$$f_1 = (\lambda) e_1 \qquad (\lambda_1) = {}_{\mathcal{B}}T_{\mathcal{E}}$$

$${}_{\mathcal{B}}T_{\mathcal{D}} = [5]$$

$\det({}_{\mathcal{B}}T_{\mathcal{E}}) = -4 \quad \therefore$ opposite orientations

$\det({}_{\mathcal{B}}T_{\mathcal{D}}) = 5 \quad \therefore$ same orientation

$\mathcal{B}, \mathcal{D}$ are same orientation & $\mathcal{E}$ has opp. orientation

Example 1.61 Let us now consider a 2-dimensional example. Consider the following two bases bases for $\mathbb{E}^2$

$$\mathcal{B} = \left(\overset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right) \qquad \mathcal{C} = \left(\overset{f_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \overset{f_2}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \right) \qquad \begin{aligned} f_1 &= e_1 + e_2 \\ f_2 &= e_1 - e_2 \end{aligned}$$

Do they have the same or opposite orientation?

$${}_{\mathcal{B}}T_{e} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \therefore \det({}_{\mathcal{B}}T_{e}) = (-1) - (1) = -2$$

$\therefore$ opposite orientation.

We shall denote "basis $\mathcal{C}$ has the same orientation as basis $\mathcal{B}$" by the notation $\mathcal{B} \sim \mathcal{C}$.

Wikipedia: Equivalence relation

Proposition 1.62 *The relation $\sim$ is an equivalence relation on the set of all bases for a given vector space. That is, $\sim$ is reflexive, symmetric and transitive.*

$\mathcal{B} \sim \mathcal{B}$ $\qquad$ $\mathcal{C} \sim \mathcal{B} \Rightarrow \mathcal{B} \sim \mathcal{C}$

Proof. $\mathcal{B} \sim \mathcal{C}, \mathcal{C} \sim \mathcal{D} \Rightarrow \mathcal{B} \sim \mathcal{D}$

Reflexivity "basis $\mathcal{B}$ has the same orientation as basis $\mathcal{B}$"

The transition matrix of $\mathcal{B}$ to itself is the identity matrix, which has positive determinant. Thus $\mathcal{B} \sim \mathcal{B}$. ∎

(det = 1)

Symmetricity

Let $\boxed{\mathcal{B} \sim \mathcal{C}}$ and T ($= {}_{\mathcal{B}}T_{e}$) be the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. The transition matrix from $\mathcal{C}$ to $\mathcal{B}$ is then T^{-1}. Now since $\det(T^{-1}) = (\det T)^{-1}$ and $\det T > 0$ we have that $\det(T^{-1}) > 0$ and so $\mathcal{C} \sim \mathcal{B}$. ∎

Transitivity

Let $\mathcal{B} \sim \mathcal{C}$ and $\mathcal{C} \sim \mathcal{D}$ and let (& want to show $\mathcal{B} \sim \mathcal{D}$)

$${}_{\mathcal{B}}T_{\mathcal{C}} \quad \text{be the transition matrix from } \mathcal{B} \text{ to } \mathcal{C}$$
$${}_{\mathcal{C}}T_{\mathcal{D}} \quad \text{be the transition matrix from } \mathcal{C} \text{ to } \mathcal{D}$$
$${}_{\mathcal{B}}T_{\mathcal{D}} \quad \text{be the transition matrix from } \mathcal{B} \text{ to } \mathcal{D}$$

We know that $\det {}_{\mathcal{B}}T_{\mathcal{C}} > 0$ and $\det {}_{\mathcal{C}}T_{\mathcal{D}} > 0$ and we need to establish that $\det {}_{\mathcal{B}}T_{\mathcal{D}} > 0$. By Lemma 1.41, ${}_{\mathcal{B}}T_{\mathcal{D}} = \boxed{{}_{\mathcal{B}}T_{\mathcal{C}}\, {}_{\mathcal{C}}T_{\mathcal{D}}}$ therefore $\det({}_{\mathcal{B}}T_{\mathcal{D}}) = \underbrace{\det({}_{\mathcal{C}}T_{\mathcal{D}})}_{\text{positive}} \underbrace{\det({}_{\mathcal{B}}T_{\mathcal{C}})}_{\text{positive}}$, a product of two positive values and so $\mathcal{B} \sim \mathcal{D}$. ∎

Since orientation is an equivalence relation this means that the set of all bases decomposes into a disjoint union of equivalence classes. Two bases of a vector space have the same orientation if and only if there are in the same equivalence class.

Example 1.60 shows that there are at least two orientation classes in a 1-dimensional vector space. In general, for a space with dimension at least 2, one can show that the determinant of the transition matrix from the basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ to the basis $(\mathbf{e}_2, \mathbf{e}_1, \ldots, \mathbf{e}_n)$, with the first two vectors swapped, has determinant -1. This

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

means that for all vector spaces there are always at least two equivalence classes of bases: a given orientation and its opposite orientation.

We want to show that these are the only two possibilities. That is, there are exactly two orientation classes.

Proposition 1.63 *Let $\mathcal{B}$ be a basis for a vector space V and let $\mathcal{B}'$ be a basis with an opposite orientation to $\mathcal{B}$.*

If $\mathcal{C}$ is a basis for V then either $\mathcal{C} \sim \mathcal{B}$ *or* $\mathcal{C} \sim \mathcal{B}'$.

Proof. Let T $= {}_{\mathcal{B}}T_{\mathcal{C}}$ be the transition matrix from basis $\mathcal{B}$ to $\mathcal{C}$ and T' $= {}_{\mathcal{B}'}T_{\mathcal{C}}$ be the transition matrix from basis $\mathcal{B}'$ to $\mathcal{C}$. Let S $\to {}_{\mathcal{B}}T_{\mathcal{B}'}$ be the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

If $\boxed{\det T' > 0}$ $\Rightarrow \mathcal{C} \sim \mathcal{B}'$ then we are done, so let us assume that $\det T' < 0$. We have seen that $T = ST'$ (Lemma 1.41) and therefore $\det T = \det T' \det S$. As $\mathcal{B}$ and $\mathcal{B}'$ have opposite orientation then $\det S$ is negative and we have assumed that $\det T'$ is also negative. This shows that $\det T > 0$ and therefore that $\mathcal{B} \sim \mathcal{C}$. ∎

$$\det(T) = \det(ST') = \det(S)\det(T') = \underbrace{\det(T')}_{<0}\underbrace{\det(S)}_{<0} > 0$$

Definition 1.64 (Orientation) An *orientation* of a vector space is an equivalence class of bases under the equivalence relation $\sim$.

Wikipedia: orientation

Note that any choice of basis $\mathcal{B}$ implicitly chooses an orientation: the equivalence class of $\mathcal{B}$ under the relation $\sim$. The proposition above tells us that there are two orientations, and that every basis has either the same orientation as a fixed given basis or the opposite orientation to it.

We may pick an orientation and call it the *left orientation* and its opposite the *right orientation*, though such a choice is arbitrary. A basis with a left orientation is sometimes referred to as a *left basis* and a basis with a right orientation is sometimes referred to as a *right basis*.

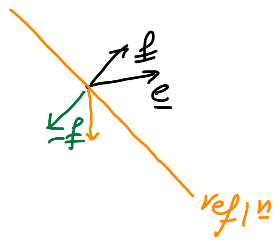
Definition 1.65 (Oriented vector space) An *oriented vector space* is a vector space together with a choice of orientation.

Example 1.66 Let $(\mathbf{e}, \mathbf{f})$ be a basis of a 2-dimensional vector space. We shall say that $(\mathbf{e}, \mathbf{f})$ has a *left orientation*.

We will consider the bases $(\mathbf{e}, -\mathbf{f})$ ①, $(\mathbf{f}, -\mathbf{e})$ ② and $(\mathbf{f}, \mathbf{e})$ ③. Which have right orientations and which have left orientations?

① $(e, f) \to (e, -f)$ $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\det(T) = -1$ ∴ opp. orientation

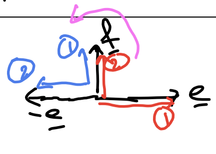
∴ $(e, -f)$ has a $\boxed{\text{right orientation}}$

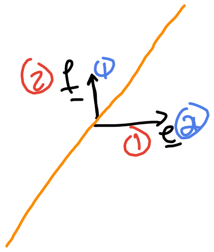


② $(e, f) \to (f, -e)$

$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\det(T) = 1$

∴ $(f, -e)$ has a $\boxed{\text{left orientation}}$





3 $(e, f) \to (f, e)$ $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\det(T) = -1$

$\therefore$ (f, e) has $\boxed{\text{right orientation}}$

Of course if we had declared our initial basis to be a right basis, then all terms left and right would have to be interchanged. The choice is entirely arbitrary.

Example 1.67 Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be a basis of $\mathbb{E}^3$ and let

$$T = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

be any 3×3 matrix with entries in $\mathbb{R}$. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be defined by

$$\mathbf{a} = a_x e_x + a_y e_y + a_z e_z$$
$$\mathbf{b} = b_x e_x + b_y e_y + b_z e_z$$
$$\mathbf{c} = c_x e_x + c_y e_y + c_z e_z$$

We have three cases:

$\det T > 0$: In this case $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis and this basis has the same orientation as $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$.

$\det T < 0$: In this case $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis and this basis has the opposite orientation to $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$.

$\det T = 0$: In this case the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are not linearly-independent and hence do not define a basis. As such they do not have an orientation.

Notice that since T was chosen arbitrarily, even if $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis it need not be orthonormal.

Remark 1.68 The important message from this section is that there exactly two orientations of any real vector space. Given a basis as a reference point, any other basis either has the same orientation or the opposite orientation as this reference point.

If two bases $\mathcal{B}$ and $\mathcal{C}$ have the same orientation then one can transform from one basis to the other via a continuous transformation. Making this statement precise is beyond the scope of this course, however we can demonstrate the point in $\mathbb{E}^3$. If $\mathcal{B}$ and $\mathcal{C}$ are orthonormal bases of $\mathbb{E}^3$ with the same orientation then there is an axis $\mathbf{v}$ such that the transformation of $\mathcal{B}$ to $\mathcal{C}$ is given by a rotation about $\mathbf{v}$. This is Euler's Theorem and will be proved in Theorem 1.77.

1.5.1 Orientation of linear operator

Let P be a linear operator acting on a vector space V and let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ be a chosen basis for V.

Consider the image of this basis $P\mathcal{B} = (P\mathbf{e}_1, P\mathbf{e}_2, \ldots, P\mathbf{e}_n)$. If P is nondegenerate then this is again a basis for V. We want to consider its orientation. As we have already seen $[P]_{\mathcal{B}}$ is the same as the transition matrix of $\mathcal{B}$ to $P\mathcal{B}$. Thus we know that if $[P]_{\mathcal{B}}$ has positive determinant then $\mathcal{B}$ and $P\mathcal{B}$ have the same orientation. If the matrix has a negative determinant then the two bases have opposite orientations .

If the determinant is zero, so P is degenerate, then $P\mathcal{B}$ is not a basis.

Since the determinant of P as a linear operator is defined to be the same as the determinant of $[P]_{\mathcal{B}}$ as a matrix we can now say the follow:

- If a linear operator P has positive determinant then the action of P preserves the orientation of a basis.
- If a linear operator P has negative determinant then the action of P swaps (changes) the orientation of a basis to the opposite orientation.

Definition 1.69 Let P be a nondegenerate (invertible) linear operator acting on a vector space V.

We say that P *preserves the orientation* of V if $\det P > 0$.

We say that P *changes the orientation* of V (swap) if $\det P < 0$

1.6 Orthogonal operators of $\mathbb{E}^n$

Recall the notion of orthogonal operator (see Subsubsection 1.4.4). In this section, we shall consider orthogonal operators in $\mathbb{E}^2$ and $\mathbb{E}^3$. In particular, we shall try to classify them and show how they relate to geometric notions that we are familiar with: rotations and reflections.

1.6.1 Orthogonal operators in $\mathbb{E}^2$

In this section, we will show that an orthogonal operator in $\mathbb{E}^2$ induces either a rotation or a reflection of $\mathbb{E}^2$, depending on whether it preserves orientation or not.

Throughout, we let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis $\mathbb{E}^2$. Recall that this implies

$(\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij})$

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 1, \quad \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 0$$

i.e., vectors $\mathbf{e}_1, \mathbf{e}_2$ have unit length and are orthogonal to each other. Also recall that by fixing an ordering on the basis, we have fixed an orientation on the basis and therefore the vector space. We shall call this the left *orientation*.

Let P be an orthogonal operator acting on $\mathbb{E}^2$, *i.e.*,

$$\langle P(\mathbf{x}),\, P(\mathbf{y})\rangle = \langle \mathbf{x},\, \mathbf{y}\rangle \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{E}^2.$$

Consider a new basis $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2)$ defined by applying P to $\mathcal{B}$:

$$\begin{aligned} \mathbf{f}_1 &= P(\mathbf{e}_1) = \alpha \mathbf{e}_1 + \gamma \mathbf{e}_2 \\ \mathbf{f}_2 &= P(\mathbf{e}_2) = \beta \mathbf{e}_1 + \delta \mathbf{e}_2 \end{aligned} \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

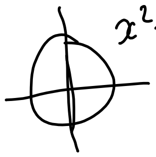
By Proposition 1.58, as P is an orthogonal operator, this new basis $\mathcal{C}$ is an orthonormal basis.

Because P is orthogonal, there are extra restriction on $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ we have not considered yet. Consider the matrix for P in the basis $\mathcal{B}$:

$$[P]_{\mathcal{B}} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

By Proposition 1.58, P is orthogonal if and only if $[P]_{\mathcal{B}}$ is an orthogonal matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = ([P]_{\mathcal{B}})^{\mathsf{T}}[P]_{\mathcal{B}} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2+\gamma^2 & \alpha\beta+\gamma\delta \\ \alpha\beta+\gamma\delta & \beta^2+\delta^2 \end{bmatrix}$$



$x^2+y^2=1$

This gives us three extra constraints on what the entries of $[P]_{\mathcal{B}}$ could be

$$\alpha^2+\gamma^2=1, \quad \beta^2+\delta^2=1, \quad \boxed{\alpha\beta+\gamma\delta=0} \tag{1.20}$$

Recall that $\alpha^2+\gamma^2=1$ and $\beta^2+\delta^2=1$ are the equations that cut out a circle of radius one. As a result, we can satisfy these equations immediately by picking angles φ, ψ and setting

$$\begin{aligned} \alpha &= \cos\varphi \qquad & \beta &= \cos\psi \\ \gamma &= \sin\varphi \qquad & \delta &= \sin\psi \end{aligned} \tag{1.21}$$

The final constraint $\alpha\beta+\gamma\delta=0$ implies that

$$\underbrace{\cos\varphi\cos\psi + \sin\varphi\sin\psi}_{\cos(\psi-\varphi)} = 0$$

$$\cos(\psi - \varphi) = 0$$

Observation 1.70 You may notice that we could have equally picked $\alpha = \sin\varphi, \gamma = \cos\varphi$ (similarly for β, δ) and still satisfied the constraints. However, this can be put in the form of equation (1.21) by picking a different angle:

$$\begin{aligned} \cos(\frac{\pi}{2} - \varphi) &= \cos(\varphi - \frac{\pi}{2}) = \cos\varphi\cos\frac{\pi}{2} + \sin\varphi\sin\frac{\pi}{2} = \sin\varphi \\ \sin(\frac{\pi}{2} - \varphi) &= -\sin(\varphi - \frac{\pi}{2}) = -\sin\varphi\cos\frac{\pi}{2} + \cos\varphi\sin\frac{\pi}{2} = \cos\varphi \end{aligned}$$

Therefore without loss of generality, we can always put $\alpha, \beta, \gamma, \delta$ in the form (1.21).

Let us pause to reinforce what we have shown so far:

Lemma 1.71 *Let $\mathcal{B}$ be an orthonormal basis and P a linear operator of $\mathbb{E}^2$. Then P is an orthogonal operator if and only if its matrix $[P]_\mathcal{B}$ can be written in the form*

$$[P]_\mathcal{B} = \begin{bmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{bmatrix}$$

and satisfies the constraint $\cos(\psi-\varphi)=0$.

The condition $\cos(\psi-\varphi)=0$ only occurs when $\psi-\varphi = \boxed{\frac{\pi}{2}+k\pi}$ for some integer $k\in\mathbb{Z}$. The geometry of P varies depending on whether k is odd or even. This splits the orthogonal operators of $\mathbb{E}^2$ into two classes:

$\psi = \varphi + \frac{\pi}{2} + k\pi$

$\therefore 2m\pi$

(1) when $\psi = \varphi + \frac{\pi}{2} + 2m\pi$ for some $m\in\mathbb{Z}$ ($k=2m$ is even),

(2) when $\psi = \varphi - \frac{\pi}{2} + 2m\pi$ for some $m\in\mathbb{Z}$ ($k=2m-1$ is odd).

$\psi = \varphi + \frac{\pi}{2} + (2m-1)\pi$

$= \varphi + \frac{\pi}{2} - \pi + 2m\pi$

Case 1: Rotations in $\mathbb{E}^2$ We first consider the case where k is even. Let $k=2m$ for some integer $m\in\mathbb{Z}$. Then $\psi = \varphi+\frac{\pi}{2}+2m\pi$ and so

$$\cos\psi = \cos\left(\varphi+\frac{\pi}{2}+2m\pi\right) = \cos\left(\varphi+\frac{\pi}{2}\right) = -\sin\varphi,$$
$$\sin\psi = \sin\left(\varphi+\frac{\pi}{2}+2m\pi\right) = \sin\left(\varphi+\frac{\pi}{2}\right) = \cos\varphi.$$

Therefore the orthogonal operator depends on a single parameter φ. To emphasise this, we denote the operator P_φ and write its matrix as

$$[P_\varphi]_\mathcal{B} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{1.22}$$

$\cos\psi$ $\sin\psi$

$\det$ $\cos^2\varphi + \sin^2\varphi = 1$

Therefore, P_φ acts on an arbitrary vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ as follows:

$$[P_\varphi]_\mathcal{B}[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\varphi - x_2\sin\varphi \\ x_1\sin\varphi + x_2\cos\varphi \end{bmatrix}$$

$$\Rightarrow P_\varphi(\mathbf{x}) = (x_1\cos\varphi - x_2\sin\varphi)\,\mathbf{e}_1 + (x_1\sin\varphi + x_2\cos\varphi)\,\mathbf{e}_2 \tag{1.23}$$

Note also that $\det P_\varphi = \det[P_\varphi]_\mathcal{B} = 1$ and so *P_φ preserves orientation*.

What is the geometric behaviour of P_φ? You may recognise the matrix (1.22) as *a rotation matrix*. Explicitly, if we apply the operator P_φ to the vector $\mathbf{x}$, it *rotates* $\mathbf{x}$ by the angle φ.

Example 1.72 Fix some orthonormal basis $\mathcal{B} = (\mathbf{e}_1,\mathbf{e}_2)$ for $\mathbb{E}^2$. Consider the vector $\mathbf{x} = 2\mathbf{e}_1+\mathbf{e}_2$ and let $\varphi = \frac{\pi}{2}$. What is the P_φ?

↳ rotate by $\pi/2$

$$(1.23) \Rightarrow P_{\frac{\pi}{2}}(\mathbf{x}) = \Big(\overset{x_1}{2}\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} - \overset{x_2}{1}\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1}\Big)\mathbf{e}_1 + \Big(\overset{x_1}{2}\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} + \overset{x_2}{1}\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0}\Big)\mathbf{e}_2$$

$= (-1)\underline{e}_1 + (2)\underline{e}_2$

$= 2\underline{e}_2 - \underline{e}_1$

You may notice that we have not addressed in this example whether the rotation acts clockwise or anti-clockwise. These two notions do not make sense in an arbitrary vector space: it is completely determined by the Orientation of the vector space and therefore the ordering of the basis. Explicitly, the rotation operator P_φ rotates the first basis vector towards the second basis vector.

Figure 1 shows Example 1.72 for two different choices of basis. In both cases, the rotation operator rotates $\mathbf{x}$ by $\frac{\pi}{2}$ radians in the direction of $\mathbf{e}_1$ to $\mathbf{e}_2$.

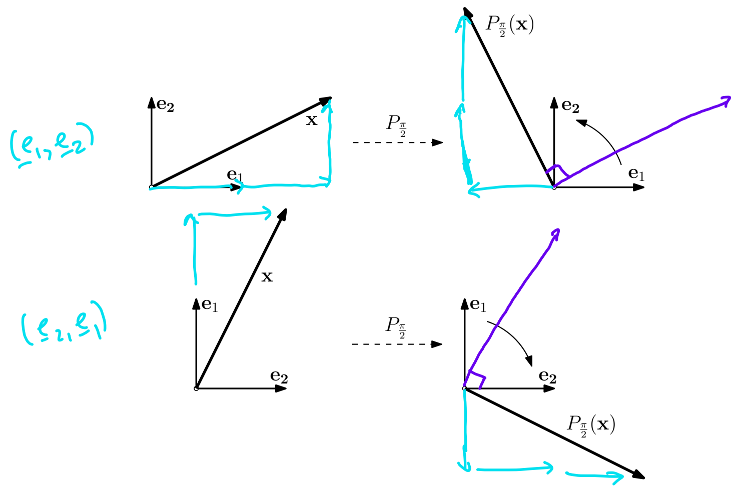


Figure 1: The rotation operator $P_{\frac{\pi}{2}}$ from Example 1.72 for two different ordered bases. The operator rotates the vector $\mathbf{x}$ by $\frac{\pi}{2}$ radians in the direction of $\mathbf{e}_1$ to $\mathbf{e}_2$.

Case 2: Reflections in $\mathbb{E}^2$ We now consider the case where k is odd. Let $k = 2m - 1$ for some integer $m \in \mathbb{Z}$. Then $\psi = \varphi - \frac{\pi}{2} + 2m\pi$ and so

$$\cos\psi = \cos\left(\varphi - \frac{\pi}{2} + 2m\pi\right) = \cos\left(\varphi - \frac{\pi}{2}\right) = \sin\varphi$$
$$\sin\psi = \sin\left(\varphi - \frac{\pi}{2} + 2m\pi\right) = \sin\left(\varphi - \frac{\pi}{2}\right) = -\cos\varphi$$

$$\begin{bmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{bmatrix}$$

Again, the orthogonal operator depends on a single variable φ. We shall denote the operator Q_φ and write its matrix as:

$$[Q_\varphi]_{\mathcal{B}} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} \tag{1.24}$$

$\cos\psi$

$\sin\psi$

By a similar calculation to (1.23), Q_φ acts on an arbitrary vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ as follows:

$$[Q_\varphi]_\mathcal{B}[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\varphi + x_2\sin\varphi \\ x_1\sin\varphi - x_2\cos\varphi \end{bmatrix}$$

$$\Rightarrow Q_\varphi(\mathbf{x}) = (x_1\cos\varphi + x_2\sin\varphi)\mathbf{e}_1 + (x_1\sin\varphi - x_2\cos\varphi)\mathbf{e}_2 \tag{1.25}$$

det
$-\cos^2\varphi - \sin^2\varphi$
$= -(\cos^2\varphi + \sin^2\varphi) = -1$

Note that in this case, $\det Q_\varphi = \det\,[Q_\varphi]_\mathcal{B} = -1$, and so Q_φ does *not* preserve orientation.

We saw that P_φ preserves orientation whereas Q_φ doesn't. How does the geometry of Q_φ compare to P_φ? To answer this, we introduce a new linear operator R on $\mathbb{E}^2$ defined as follows:

$$R(\mathbf{e}_1) = \mathbf{e}_1\,,\; R(\mathbf{e}_2) = -\mathbf{e}_2 \quad \Rightarrow \quad [R]_\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

R is the reflect that sends $\mathbf{e}_2$ to $-\mathbf{e}_2$, or alternatively is the reflection in the line spanned by $\mathbf{e}_1$. Recall from Example 1.66 that this operator does not preserve orientation. Furthermore, by comparing $[P_\varphi]_\mathcal{B}$ and $[Q_\varphi]_\mathcal{B}$ we can deduce that Q_φ is *the composition of a rotation and a reflection.*

$$[Q_\varphi]_\mathcal{B} = \underbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}} = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{R} = [P_\varphi]_\mathcal{B}[R]_\mathcal{B}. \tag{1.26}$$

Example 1.73 Similar to the previous example, we fix some orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ for $\mathbb{E}^2$ and consider the vector $\mathbf{x} = 2\mathbf{e}_1 + \mathbf{e}_2$. Let $\varphi = 0$. What is Q_0?

Intuitively Q_0 should be R.

$$Q_0(\underline{x}) = \big(\overset{x_1}{2}\underbrace{\cos(\overset{\varphi}{0})}_{1} + \overset{x_2}{1}\underbrace{\sin(\overset{\varphi}{0})}_{0}\big)\,\underline{e}_1 + \big(\overset{x_1}{2}\underbrace{\sin(\overset{\varphi}{0})}_{0} - \overset{x_2}{1}\underbrace{\cos(\overset{\varphi}{0})}_{1}\big)\,\underline{e}_2$$

$$= 2\underline{e}_1 - \underline{e}_2$$

$$Q_{\frac{\pi}{2}}(\underline{x}) = \big(2\underbrace{\cos(\tfrac{\pi}{2})}_{0} + \underbrace{\sin(\tfrac{\pi}{2})}_{1}\big)\,\underline{e}_1 + \big(2\underbrace{\sin(\tfrac{\pi}{2})}_{1} - \underbrace{\cos(\tfrac{\pi}{2})}_{0}\big)\,\underline{e}_2$$

$$= \underline{e}_1 + 2\underline{e}_2$$

2 geometric intuitions of $Q_{\pi/2}$

(1) It's the refl^n of the line spanned by $\underline{e}_1$ followed by a rot^n $\frac{\pi}{2}$

(2) It's the refl^n in the line spanned by $\underline{e}_1 + \underline{e}_2$. (fig 2)

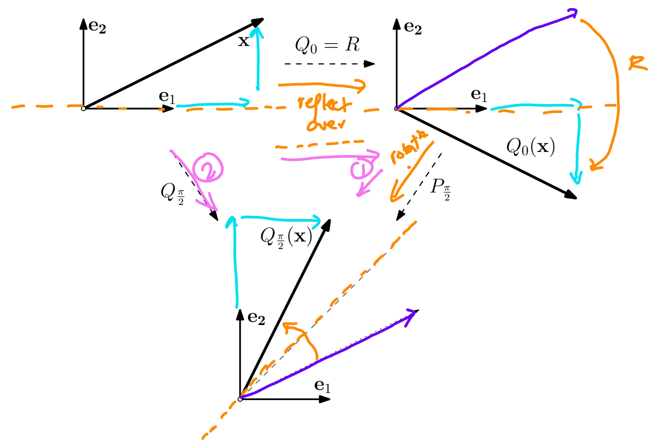


Figure 2: The reflections $Q_0 = R$ and $Q_{\frac{\pi}{2}}$ applied to the vector $\mathbf{x}$ from Example 1.73. Note that $Q_{\frac{\pi}{2}}$ can be viewed as the reflection in the line $\mathbf{e}_1 + \mathbf{e}_2$, or as the composition of R and $P_{\frac{\pi}{2}}$.

This completes a full characterisation of orthogonal operators in $\mathbb{E}^2$. We recall what we have shown in the following proposition.

Proposition 1.74 *Let P be an arbitrary orthogonal linear operator on $\mathbb{E}^2$, then* $\det P = \pm 1$.

If $\det P = 1$ *then there exists an angle* $\varphi \in [0, 2\pi)$ *such that* $P = P_\varphi$ *is the operator that* rotates *a vector by* φ.

If $\det P = -1$ *then there exists an angle* $\varphi \in [0, 2\pi)$ *such that* $P = Q_\varphi$ *is the operator that* rotation (2) *a vector by* φ *composed with a* reflection (1).