Welcome :)

MATH2022
Intro to Geometry

Week 1

1 Euclidean vector spaces

To begin doing geometry properly, we will need to recall quite a few notions from linear algebra. Note that we will be recalling many results from linear algebra, and therefore will not repeat proofs.

1.1 Vector spaces and basis vectors

1.1.1 Definition of a vector space

We denote the set of real numbers by $\mathbb{R}$. Informally, we think of a vector space as a set of vectors such that

- adding two vectors together gives us another vector in the vector space,
- multiplying a vector by a real number (or *scalar*) gives another vector in the vector space.

vectors space addition

scalar mult

Definition 1.1 (Vector space) A *vector space* (over $\mathbb{R}$) $(V, +, \cdot)$ is a set of vectors, along with an addition operation $+$ and a multiplication operation $\cdot$ satisfying the following axioms for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $\lambda, \mu \in \mathbb{R}$:

vectors

scalars

- (Additive closure) $a + b \in V$,
- (Additive commutativity) $a + b = b + a$,
- (Additive associativity) $(a + b) + c = a + (b + c)$
- (Zero) $\exists \mathbf{0} \in V$ such that $\forall \mathbf{a} \in V$, $a + 0 = a$,
- (Additive inverses) $\exists \bar{a}' \in V$ s.t. $a + a^{-1} = 0$
- (Multiplicative closure) $\lambda \cdot a \in V$,
- (Multiplicative associativity) $(\lambda\mu) \cdot a = \lambda \cdot (\mu \cdot a)$
- (Distributivity) $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$
- (Distributivity) $(\lambda + \mu) \cdot a = \lambda a + \mu a$

While we shall only be concerned with vector spaces over the real numbers in this course, in general vector spaces can have scalars in any field, e.g. $\mathbb{Q}, \mathbb{C}, \mathbb{F}_p$, etc. Wikipedia: vector space

(for $\mathbb{R}$ normally we take scalar
$a \in \mathbb{R}$, $a^{-1} \in \mathbb{R}$
$a^{-1} = -a$
$a + a^{-1} = a + (-a) = 0$

- (Unity) $1 \cdot \underline{a} = \underline{a}$.

Remark 1.2 We will denote vectors by writing them bold in the notes, or underlining them when writing. This is particularly necessary when we have to distinguish between 0, the real number, and $\mathbf{0}$, the zero element of the vector space.

Example 1.3 The most natural example of a vector space (over real numbers) is the space of ordered n-tuples of real numbers:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

transpose

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n)^\mathsf{T} \mid x_1, \ldots, x_n \in \mathbb{R}\} \tag{1.1}$$

where $(-)^\mathsf{T}$ denotes the transpose.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two vectors where for $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$, $\mathbf{y} = (y_1, \ldots, y_n)^\mathsf{T}$, we define addition as

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^\mathsf{T}$$

and multiplication by scalars $\lambda \in \mathbb{R}$ as

$$\lambda \cdot \mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$$

Remark 1.4 When defining this vector space we have a choice between row vectors and column vectors. Here, and throughout these notes, we will use the convention of column vectors; this is chosen so that matrices multiply on the left of vectors: $M\mathbf{x}$, mirroring the idea of functions being denoted on the left: $f(x)$.

1.1.2 Linear dependence

As vector spaces are closed under addition and scalar multiplication, we will often want to consider *linear combinations* of vectors:

scalar mult

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}_i = \boxed{\lambda_1 \mathbf{x}_1} + \lambda_2 \mathbf{x}_2 + \cdots + \boxed{\lambda_m \mathbf{x}_m}, \tag{1.2}$$

vector

where $\lambda_i \in \mathbb{R}$ are scalars (real numbers) and $\mathbf{x}_i \in V$ are vectors from the vector space V.

Definitions 1.5 (Linear dependence and independence) The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exist m scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ (not all equal to zero) such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \cdots + \lambda_m \underline{x}_m = \underline{0} \tag{1.3}$$

$$\Rightarrow \lambda_1 \underline{x}_1 + \cdots + \lambda_{i-1} \underline{x}_{i-1} + \lambda_{i+1} \underline{x}_{i+1} + \cdots + \lambda_m x_m = -\lambda_i \underline{x}_i$$

If the vectors are not linearly dependent, we say they are *linearly independent.*

Wikipedia: linearly independent

Note We have to demand not all of our scalars be equal to zero, otherwise (1.3) is true for *every* set of vectors.

There are a couple of equivalent definitions of linear independence and dependence that can be more useful in practice. A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ are linearly independent if and only if

($\lambda_1 \underline{x}_1 + \cdots + \lambda_m \underline{x}_m = 0$ if & only if $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$)

The following proposition gives a useful definition of linear dependence.

Proposition 1.6 *The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ are linearly dependent if and only if at least one of these vectors can be expressed as a linear combination of other vectors*

$$\boxed{\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j, \quad \lambda_j \in \mathbb{R}}$$ (see (1.3))

1.1.3 Basis and dimension of a vector space

As vector spaces are closed under addition of vectors, we would like to find a small set of vectors such that we can write any vector as a linear combination of vectors in this set. This leads to the notion of a *basis*.

Given a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in V$, we define their *span* to be the set of all vectors that can be written as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$, *i.e.*,

Wikipedia: span

$$\operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \left\{ \underline{y} \in V \mid \underline{y} = \sum_{i=1}^{m} \lambda_i \underline{x}_i, \ \lambda_i \in \mathbb{R} \right\} \tag{1.4}$$

We say the vectors *span V* if $\operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = V$.

Definitions 1.7 (Basis and ordered basis) A set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset V$ form a *basis* of V if ① they span V & ② they are lin. indep. A tuple of vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ form an *ordered basis* if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis as a set.

Wikipedia: basis

Example 1.8 Let $V = \mathbb{R}^2$ be a vector space and suppose we have the following two vectors:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Show that they form a basis.

② linear independence.

$\lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ what are λ_1, λ_2 = ?

$$\underline{\lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2} = \begin{bmatrix} \lambda_1 \cdot 1 \\ \lambda_1 \cdot 1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \cdot 0 \\ \lambda_2 \cdot 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 + 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \lambda_1 = 0$

$\therefore$ lin. ind.

$\lambda_1 + 2\lambda_2 = 0$
$0 + 2\lambda_2 = 0 \Rightarrow \lambda_2 = 0$

② span V?

$$\lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2 = \begin{bmatrix} x \\ y \end{bmatrix} \in V = \mathbb{R}^2$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + 2\lambda_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \to \lambda_1 = x, \quad \to x + 2\lambda_2 = y \Rightarrow \lambda_2 = \frac{y-x}{2}$$

$\therefore \{e_1, e_2\}$ is a basis of V

The difference between basis and ordered basis is a subtle, but important one. In a tuple, the order in which we list the element matters, whereas order does not matter as a set. For example, the sets $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_2, \mathbf{e}_1\}$ are the same, whereas the tuples $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_2, \mathbf{e}_1)$ are *not* the same.

The following theorem highlights why bases are hugely important objects for describing vector spaces.

Theorem 1.9 *A set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ form a basis for V if and only if any vector $\mathbf{x} \in V$ can be expressed* uniquely *as a linear combination*

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i, \; \lambda_i \in \mathbb{R}$$

Remark 1.10 The most important word in this theorem is Uniquely: there is precisely one expression for $\mathbf{x}$. This does not hold if our vectors are not linearly independent. If we take a set of vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ that span V, we can express any $\mathbf{x} \in V$ as some linear combination

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}_i = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_m \mathbf{x}_m = \mathbf{x}.$$

However, if $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ don't form a basis, they must be linearly dependent and so there exists some choice of (non-zero) scalars $\mu_i \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} \mu_i \mathbf{x}_i = \mu_1 \mathbf{x}_1 + \cdots + \mu_m \mathbf{x}_m = \mathbf{0}.$$

Now we can deduce $\sum_{i=1}^{m} \lambda_i \mathbf{x}_i$ is not a unique expression for $\mathbf{x}$, as we can write

$$\underline{x} = \underline{x} + \underline{0} = \overbrace{\lambda_1 \underline{x}_1 + \cdots + \lambda_m \underline{x}_m}^{\underline{x}} + \overbrace{\mu_1 \underline{x}_1 + \cdots + \mu_m \underline{x}_m}^{\underline{0}}$$
$$= \underbrace{(\lambda_1 + \mu_1)} \underline{x}_1 + \cdots + (\lambda_m + \mu_m) \underline{x}_m$$

We say "a basis" of a vector space rather than "the basis" as a vector space has many different bases. However, every basis of a given vector space has the same size: if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ are both a basis of V, then $n = m$. This invariance is what leads to the notion of *dimension*.

Definition 1.11 (Dimension) The *dimension* of a vector space V is the size of a basis for V.

Wikipedia: dimension

Example 1.12 (The canonical basis of $\mathbb{R}^n$) Recall from (1.1) the vector space

$$\mathbb{R}^n = \left\{ (x_1, x_2, \dots, x_n)^\mathsf{T} \mid x_1, \dots, x_n \in \mathbb{R} \right\}.$$

Consider the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \ \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{1.5}$$

One can check that these vectors span $\mathbb{R}^n$ and are linearly independent, therefore they form a basis for $\mathbb{R}^n$. For any vector $\mathbf{a} = (a_1, \dots, a_n)^\mathsf{T} \in \mathbb{R}^n$, we can express it as a linear combination of

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$a = (a_1, \dots, a_n)^T$

$\mathbf{e}_1, \dots, \mathbf{e}_n$ as follows:

$$\mathbf{a} = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + \cdots + a_n \underline{e}_n$$

Wikipedia: canonical basis

This basis is called the *canonical basis* of $\mathbb{R}^n$: it is not the only choice of basis, but it is the most natural due to how simple it is to express vectors in it.

Remark 1.13 While all vector spaces we consider will be finite dimensional, some vector spaces have a basis that consists of infinitely many basis vectors. An example of an infinite-dimensional vector space is the space of polynomials in one variable,

$$\mathbb{R}[x] = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \mid a_i \in \mathbb{R}\}$$

This is indeed a vector space and has the basis $\{1, x, x^2, x^3, \dots, x^N, \dots\}$. There are infinitely many elements in this basis, therefore $\mathbb{R}[x]$ is infinite-dimensional.

Exercise 1.14 Check $\mathbb{R}[x]$ is a vector space and $\{1, x, x^2, x^3, \dots\}$ is a basis - do not be put off by the word infinite!

1.1.4 Change of basis and transition matrices

Vector spaces do not have a unique choice of basis, there are many choices of basis. Furthermore, if we need to change from one basis to another, we would like to do it in a controlled way. This is where *transition matrices* are useful.

Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ be an arbitrary *ordered* basis of an n-dimensional vector space V. Once this basis is fixed, we can represent any vector $\mathbf{x} \in V$ as a column vector, or an $n \times 1$ matrix, in the following way:

Not necessarily true!

$$\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

e_i is some arbitrary vector.

$$V \ni \underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \cdots + x_n \underline{e}_n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Note that it is very important that the basis be ordered to do this, as the ordering is what determines which entry corresponds to which basis vector.

Suppose we have an ordered set of n vectors $\mathcal{C} = (\mathbf{f}_1, \dots, \mathbf{f}_n)$ of V. As $\mathcal{B}$ is a basis, we can write $\mathbf{f}_i$ as

are vectors.

$$\mathbf{f}_i = a_{1,i}\mathbf{e}_1 + a_{2,i}\mathbf{e}_2 + \cdots + a_{n,i}\mathbf{e}_n = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{n,i} \end{bmatrix} \qquad a_{k,i} \in \mathbb{R}$$

we can do this $\forall i$

We can concatenate these n column vectors together to get an $n \times n$ matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$:

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{array}{c} \begin{array}{cccc} \mathbf{f}_1 & \mathbf{f}_2 & & \mathbf{f}_n \end{array} \\ \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \end{array} $$

The matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ tells us how to represent the ordered set of vectors $\mathcal{C}$ in terms of the basis $\mathcal{B}$.

We would like to know when this new set of vectors $\mathcal{C}$ forms an ordered basis. The following proposition states we can find this out using only properties of ${}_{\mathcal{B}}T_{\mathcal{C}}$.

Proposition 1.15 *Let $\mathcal{B}$ be an ordered basis for the n-dimensional vector space V. An ordered set of n vectors $\mathcal{C}$ form an ordered basis of V if and only if*

- ${}_{\mathcal{B}}T_{\mathcal{C}}$ has rank n
- $\det({}_{\mathcal{B}}T_{\mathcal{C}}) \neq 0$
- ${}_{\mathcal{B}}T_{\mathcal{C}}$ is invertible

equivalent ways of saying the same thing.

Sketch of proof. This proposition is a consequence of the rank-nullity theorem from linear algebra. Suppose the rank of ${}_{\mathcal{B}}T_{\mathcal{C}}$ is less than n; by the rank-nullity theorem this occurs if and only if the nullity of ${}_{\mathcal{B}}T_{\mathcal{C}}$ is greater than zero. This is equivalent to there being a nonzero vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^{\mathsf{T}}$ in the kernel of ${}_{\mathcal{B}}T_{\mathcal{C}}$, *i.e.*,

$${}_{\mathcal{B}}T_{\mathcal{C}}\boldsymbol{\lambda} = \lambda_1\mathbf{f}_1 + \cdots + \lambda_n\mathbf{f}_n = \mathbf{0}.$$

This gives a linear dependence on $(\mathbf{f}_1, \ldots, \mathbf{f}_n)$, therefore they do not form a basis. □

Recall that a matrix M such that $\det(M) \neq 0$ is called *nondegenerate* or *nonsingular*.

Wikipedia: nondegenerate

Definition 1.16 (Transition matrix) The $n \times n$ nonsingular matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ that describes an ordered basis $\mathcal{C} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)$ in terms of the ordered basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is called the *transition matrix* from $\mathcal{B}$ to $\mathcal{C}$.

The transition matrix is also referred to as a *change of basis matrix*.
Wikipedia: transition matrix

Example 1.17 Consider the matrix

$$T = \begin{bmatrix} 1 & 3 & 7 \\ 0 & \lambda & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

where $\lambda \in \mathbb{R}$ is an arbitrary parameter. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an ordered basis of V. What ordered vectors does T map this ordered basis to? Is it an ordered basis as well?

$$(\underline{e}_1 \ , \ 3\underline{e}_1 + \underset{?}{\lambda}\underline{e}_2 \ , \ 7\underline{e}_1 + 5\underline{e}_2 + 3\underline{e}_3) = \mathcal{C}$$

By prop$^{\underline{n}}$ 1.15 $\det(T) = 3\lambda \neq 0$ $\therefore$ $\mathcal{C}$ is an ordered basis if & only if $\lambda \in \mathbb{R}\setminus\{0\}$.

Remark 1.18 We stress the importance of considering $\mathcal{B}$ and $\mathcal{C}$ as ordered bases: without the ordering, we do not know which vectors of $\mathcal{B}$ get mapped to $\mathcal{C}$. For example, the ordered bases $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{C} = (\mathbf{e}_2, \mathbf{e}_1)$ of a 2-dimensional vector space V are equal as bases, but not as ordered bases. The transition matrix that reverses the order is

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

if $\mathcal{B}$ & $\mathcal{C}$ were the same then ${}_{\mathcal{B}}T_{\mathcal{C}}$ would be the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

As ordering is key for working with transition matrices (and other matrices later on), we will only consider ordered bases for the remainder of the course. Therefore we shall mostly drop the prefix "ordered". We shall continue to use round brackets to denote a tuple or ordered set of basis vectors, rather than a set.

1.2 Euclidean vector spaces

1.2.1 Inner products

Much of the geometry we learn at school is centred around distances and angles. We can give vector spaces some additional structure so that we have some notion of distance and angle. The key to this is an *inner product*.

There is a more general definition of inner products for vector spaces over other fields. As we will only be interested in real vector spaces, we'll stick with this definition.
Wikipedia: inner product

Definition 1.19 (Inner product) An *inner product* (or scalar product) on a vector space V (over $\mathbb{R}$) is a function

$$\langle -, - \rangle : V \times V \to \mathbb{R}$$

$$\underline{v}, \underline{w} \in V \quad \langle \underline{v}, \underline{w} \rangle \mapsto x \in \mathbb{R}$$

that maps two vectors to a scalar satisfying the following conditions for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and for all $\lambda, \mu \in \mathbb{R}$:

$\lambda\langle x, z\rangle \neq \langle \lambda x, \lambda z\rangle$
$\parallel$
$\langle \lambda x, z\rangle = \langle x, \lambda z\rangle$

- (symmetry) $\langle x, y\rangle = \langle y, x\rangle$,
- (linearity) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$
- (positive-definite) $\langle x, x\rangle \geq 0$ w/ equality iff $x = 0$

Definition 1.20 (Euclidean vector space) A *Euclidean vector space* is a vector space over $\mathbb{R}$ equipped with an inner product.

The connection between inner products and notions of distance and angle is emphasised in the following example.

Example 1.21 The vector space $\mathbb{R}^n$ can be viewed as a Euclidean vector space via the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1 y_1 + \cdots + x_n y_n \tag{1.6}$$

This is sometimes called the *canonical inner product*, or the *dot product*. Wikipedia: dot product

Exercise 1.22 Check that the canonical inner product is an inner product.

Example 1.23 Consider a 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. We define the inner product $\langle -, -\rangle$ such that $\langle \mathbf{e}_1, \mathbf{e}_1\rangle = 3$, $\langle \mathbf{e}_2, \mathbf{e}_2\rangle = 5$ and $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = 0$. What is the inner product for any two vectors?

symmetry — $\boxed{\langle x, y\rangle = \langle y, x\rangle}$
linearity — $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$

$x = x_1 e_1 + x_2 e_2$
$y = y_1 e_1 + y_2 e_2$

$$\begin{aligned}\langle x, y\rangle &= \langle x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2\rangle \\ &= x_1\langle e_1, y_1 e_1 + y_2 e_2\rangle + x_2\langle e_2, y_1 e_1 + y_2 e_2\rangle \\ &= x_1 y_1 \underbrace{\langle e_1, e_1\rangle}_{3} + \cancel{x_1 y_2} \underbrace{\cancel{\langle e_1, e_2\rangle}}_{0} + \cancel{x_2 y_1} \underbrace{\cancel{\langle e_2, e_1\rangle}}_{=\langle e_1, e_2\rangle = 0} + x_2 y_2 \overbrace{\cancel{\langle e_2, e_2\rangle}}^{5} \\ &= \boxed{3x_1 y_1 + 5x_2 y_2}\end{aligned}$$

Example 1.24 Consider the following nonexample of an inner product defined on V. We define $\langle -, -\rangle$ such that $\langle \mathbf{e}_1, \mathbf{e}_1\rangle = \langle \mathbf{e}_2, \mathbf{e}_2\rangle = 0$ and $\langle \mathbf{e}_1, \mathbf{e}_2\rangle = 1$. Is this an inner product?

$\langle x, y\rangle = x_1 y_2 + x_2 y_1$ $\quad\left(= x_1 y_1\langle e_1, e_1\rangle + x_1 y_2\langle e_1, e_2\rangle + \cdots \right)$

$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{matrix} x_1, y_1 \\ x_2\ y_2 \end{matrix}$ $\quad\langle x, x\rangle = 1\cdot(-1) + (-1)\cdot 1 = (-1)(-1) = -2$

(want ≥ 0)

1.2.2 Geometry of Euclidean vector spaces

We'll now use the definition of inner product to reconstruct certain geometric concepts of Euclidean vector spaces. Throughout we let V be an n-dimensional Euclidean vector space.

Let $\mathbf{x} \in V$, we define the *length* (or *magnitude*) of $\mathbf{x}$ to be

$$\|\mathbf{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}.$$

We remark that the formula for length does not depend on a choice of basis: we can change basis freely and the length of a vector stay the same. Note that when $\langle -, - \rangle$ is the dot product, our definition of length agrees with the standard definition of length from Euclidean geometry, *i.e.*,

$$\|x\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \sqrt{\underline{x} \cdot \underline{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Wikipedia: norm
Wikipedia: Euclidean norm

The map $\|\cdot\| : V \to \mathbb{R}$ is an example of a *norm*, a way of defining distance on a vector space. Length is called the *Euclidean norm* as it is the norm associated with Euclidean space. However, there are many more ways of defining distance on a space, and so there are many more norms one can define.

We can also use the inner product to define the *angle* between two vectors $\mathbf{x}, \mathbf{y} \in V$. Explicitly, the angle θ between $\mathbf{x}, \mathbf{y}$ is defined by

$$\cos(\theta) = \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \tag{1.7}$$

only thing with sign.

angle

In particular, we can use the inner product to quickly infer general behaviour about the angle:

- $\langle \mathbf{x}, \mathbf{y} \rangle > 0$ if and only if θ is acute,
- $\langle \mathbf{x}, \mathbf{y} \rangle < 0$ if and only if θ is obtuse,
- $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\mathbf{x}, \mathbf{y}$ are *orthogonal*.

don't depend on θ

Wikipedia: orthogonal
In the case of Euclidean vector spaces, two vectors are orthogonal if and only if they are *perpendicular*: the angle between them is a right angle. Orthogonality is a generalisation of perpendicularity to any vector space.
Wikipedia: Perpendicular

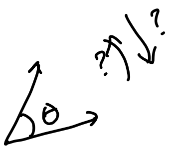
This last case will be of particular importance in the following section.

Similarly to length, setting $\langle -, - \rangle$ to the dot product recovers the identity

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

and so our definition of angle agrees with the standard definition from Euclidean geometry.

Remark 1.25 As $\cos(\theta) = \cos(-\theta)$, the inner product only defines angles up to sign. The intuition behind this is there is no good notion of what is a positive or negative angle. You may *choose* for angles to be measured "clockwise", but there is no good reason why



you shouldn't choose "anti-clockwise". This issue will be dealt with in Subsection 1.5 when we discuss the orientation of a vector space.