Week 11/12

5.3 Collinearity

In this section we consider what is meant by points in projective space being *collinear*. Intuitively we want to say that if points are collinear in affine coordinates then they are collinear in projective space. Let us consider the projective plane $\mathbb{P}^2$, with points $[x : y : z]$ and the affine chart with non-zero final coordinate, with affine coordinates $(\frac{x}{z}, \frac{y}{z}) = (u, v)$. The general equation of a line in $\mathbb{A}^2$ is given by $au + bv + c = 0$ for some real values a, b and c. A point $[x : y : z]$ has affine coordinates on this line if

which is true if

This second equation is the general equation for a plane in $\mathbb{A}^3$ that passes through the origin. The same ideas hold in higher dimensions: three points will be collinear in affine coordinates if the homogeneous coordinates represent points on the same plane.

Wikipedia: collinear

Definition 5.15 (Collinearity) We say that three points, $[x_1 : x_2 : \cdots : x_{n+1}]$, $[y_1 : y_2 : \cdots : y_{n+1}]$ and $[z_1 : z_2 : \cdots : z_{n+1}]$ in $\mathbb{P}^n$ are *collinear*, or that they lie on the same projective line, if there is a plane ________________________ in $\mathbb{A}^{n+1}$ that contains the points $(x_1, \ldots, x_{n+1})$, $(y_1, \ldots, y_{n+1})$ and $(z_1, \ldots, z_{n+1})$.

Recall that three arbitrary vectors support points on the same plane if they are linearly dependent, this gives an alternative way to calculate collinearity in $\mathbb{P}^2$. Three points $[x_1 : y_1 : z_1]$, $[x_2 : y_2 : z_2]$ and $[x_3 : y_3 : z_3]$ are collinear if their associated vectors are linearly dependent; or equivalently when

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = 0.$$

Example 5.16 Consider the points

$$\mathbf{A} = [0 : -1 : 2],\ \mathbf{B} = [-1 : 0 : 3],\ \mathbf{C} = [2 : -4 : 2].$$

Are they colinear?

Consider now the point $\mathbf{D} = [2 : -3 : 0]$. Are $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ colinear?

5.4 The cross ratio

Recall that the property preserved by affine transformations was the ratio of lengths of collinear line segments. This means that if T is an affine transformation and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are collinear points then

$$\frac{\mathbf{A}-\mathbf{C}}{\mathbf{B}-\mathbf{C}} = \frac{T(\mathbf{A})-T(\mathbf{C})}{T(\mathbf{B})-T(\mathbf{C})}. \tag{5.3}$$

When we generalize to projective transformations, it is no longer true that this ratio is preserved (see Example 5.21). The equivalent property that is preserved for projective transformations is called the *cross ratio* and involves a fourth point.

Definition 5.17 (Cross ratio) Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be four collinear points in affine space $\mathbb{A}^n$. The *cross ratio* $(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ is defined to be

Wikipedia: cross ratio

$$(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \tag{5.4}$$

By extension, the cross ratio of four collinear points in projective space is the cross ratio of the four points in any affine chart.

Remark 5.18 Note that Definition 5.17 and equation (5.3) do not make sense if the points are not collinear as we cannot divide arbitrary vectors. This is only well defined when the vectors are parallel and therefore multiples of one another.

Example 5.19 Consider the following four points in $\mathbb{P}^2$:

$$\mathbf{A} = [1 : 0 : 0],\ \mathbf{B} = [1 : 1 : 1],\ \mathbf{C} = [1 : 2 : 2],\ \mathbf{D} = [1 : 3 : 3].$$

What is their cross-ratio?

It is not immediately clear that the cross ratio is well-defined for projective points, however the following proposition deals with this issue.

Proposition 5.20 *Let* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$ *and* $\mathbf{D}$ *be four collinear points in* $\mathbb{P}^n$ *and let* $T\colon \mathbb{P}^n \to \mathbb{P}^n$ *be a projective transformation. Then the cross ratio of the four points is preserved by* T*:*

$$(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) =$$

In particular, the cross-ratio is independent of the choice of affine chart.

Proof. We prove the result for $\mathbb{P}^1$.

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be (collinear) points of $\mathbb{P}^1$ and let $T\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the projective transformation:

$$[x : y] \overset{T}{\longmapsto} [\alpha x + \beta y : \gamma x + \delta y].$$

After possibly changing basis we can assume that all the points lie in the second affine chart and we have the map of affine coordinates

$$u \overset{T}{\longmapsto} \frac{\alpha u + \beta}{\gamma u + \delta}.$$

Let $u_{\mathbf{A}}$, $u_{\mathbf{B}}$, $u_{\mathbf{C}}$ and $u_{\mathbf{D}}$ be the affine coordinates of the points in this affine chart. The cross ratio of the images of the points is now

$$\begin{aligned}
&(T(\mathbf{A}), T(\mathbf{B}); T(\mathbf{C}), T(\mathbf{D})) \\
&= \frac{\left(\frac{\alpha u_{\mathbf{A}}+\beta}{\gamma u_{\mathbf{A}}+\delta} - \frac{\alpha u_{\mathbf{C}}+\beta}{\gamma u_{\mathbf{C}}+\delta}\right)\left(\frac{\alpha u_{\mathbf{B}}+\beta}{\gamma u_{\mathbf{B}}+\delta} - \frac{\alpha u_{\mathbf{D}}+\beta}{\gamma u_{\mathbf{D}}+\delta}\right)}{\left(\frac{\alpha u_{\mathbf{B}}+\beta}{\gamma u_{\mathbf{B}}+\delta} - \frac{\alpha u_{\mathbf{C}}+\beta}{\gamma u_{\mathbf{C}}+\delta}\right)\left(\frac{\alpha u_{\mathbf{A}}+\beta}{\gamma u_{\mathbf{A}}+\delta} - \frac{\alpha u_{\mathbf{D}}+\beta}{\gamma u_{\mathbf{D}}+\delta}\right)} \\
&= \frac{\left(\frac{(\alpha u_{\mathbf{A}}+\beta)(\gamma u_{\mathbf{C}}+\delta)-(\alpha u_{\mathbf{C}}+\beta)(\gamma u_{\mathbf{A}}+\delta)}{(\gamma u_{\mathbf{A}}+\delta)(\gamma u_{\mathbf{C}}+\delta)}\right)\left(\frac{(\alpha u_{\mathbf{B}}+\beta)(\gamma u_{\mathbf{D}}+\delta)-(\alpha u_{\mathbf{D}}+\beta)(\gamma u_{\mathbf{B}}+\delta)}{(\gamma u_{\mathbf{B}}+\delta)(\gamma u_{\mathbf{D}}+\delta)}\right)}{\left(\frac{(\alpha u_{\mathbf{B}}+\beta)(\gamma u_{\mathbf{C}}+\delta)-(\alpha u_{\mathbf{C}}+\beta)(\gamma u_{\mathbf{B}}+\delta)}{(\gamma u_{\mathbf{B}}+\delta)(\gamma u_{\mathbf{C}}+\delta)}\right)\left(\frac{(\alpha u_{\mathbf{A}}+\beta)(\gamma u_{\mathbf{D}}+\delta)-(\alpha u_{\mathbf{D}}+\beta)(\gamma u_{\mathbf{A}}+\delta)}{(\gamma u_{\mathbf{A}}+\delta)(\gamma u_{\mathbf{D}}+\delta)}\right)} \\
&= \frac{\left(\frac{(\alpha\delta-\beta\gamma)(u_{\mathbf{A}}-u_{\mathbf{C}})}{(\gamma u_{\mathbf{A}}+\delta)(\gamma u_{\mathbf{C}}+\delta)}\right)\left(\frac{(\alpha\delta-\beta\gamma)(u_{\mathbf{B}}-u_{\mathbf{D}})}{(\gamma u_{\mathbf{B}}+\delta)(\gamma u_{\mathbf{D}}+\delta)}\right)}{\left(\frac{(\alpha\delta-\beta\gamma)(u_{\mathbf{B}}-u_{\mathbf{C}})}{(\gamma u_{\mathbf{B}}+\delta)(\gamma u_{\mathbf{C}}+\delta)}\right)\left(\frac{(\alpha\delta-\beta\gamma)(u_{\mathbf{A}}-u_{\mathbf{D}})}{(\gamma u_{\mathbf{A}}+\delta)(\gamma u_{\mathbf{D}}+\delta)}\right)} \\
&= \frac{(u_{\mathbf{A}} - u_{\mathbf{C}})(u_{\mathbf{B}} - u_{\mathbf{D}})}{(u_{\mathbf{B}} - u_{\mathbf{C}})(u_{\mathbf{A}} - u_{\mathbf{D}})} \\
&= (\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}).
\end{aligned}$$

In higher dimensions we can use the fact that collinear points are all in 1-dimensional subspace equivalent to $\mathbb{P}^1$ and use the proof above. This makes sense geometrically, however to make this rigorous would require more technically details than we wish to cover. □

The alternative approach in higher dimensions is to rearrange some even more tedious equations, which the author (and likely the reader) would rather avoid doing.

Example 5.21 Consider the projective transformation $T\colon \mathbb{P}^2 \to \mathbb{P}^2$ given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Does the projective transofmration preserve ratio of distances? Does it preserve the cross ratio?

5.5 Projective transformations of conic sections

We proved in Theorem 4.20 that affine transformations map ellipses to ellipses, hyperbolas to hyperbolas and parabolas to parabolas, but that we cannot map one type of conic section onto another. Projective transformations do not have this limitation. In this section we will prove that we ______ map different types of conic section onto one another via projective transformations.

Consider the set of points in the projective plane

$$D = \left\{[x : y : z] \in \mathbb{P}^2 \ \mid \ x^2 + y^2 = z^2\right\} \subset \mathbb{P}^2.$$

When restricted to the third affine chart, D just becomes the unit circle:

$$D \cap \mathbb{A}^2_3 = \qquad\qquad\qquad\qquad\qquad\qquad \subset \mathbb{A}^2.$$

The following theorem shows we can map the unit circle to any other conic section given the correct choice of projective transformation.

Theorem 5.22 *For any conic section $C \subset \mathbb{A}^2$, there exists a projective transformation $T\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that C is $T(D)$ restricted to the ________ affine chart.*

Proof. We split this into cases for each type of conic section.

Ellipse: The case of an ellipse has already been covered. A special case of Proposition 4.16 shows that the affine transformation

$$(u, v) \mapsto (au, bv)$$

maps the unit circle to the ellipse with equation $\frac{u^2}{v^2} + \frac{y^2}{b^2} = 1$. As a projective transformation, this is

$$[x : y : z] \mapsto [ax : by : z].$$

Hyperbola: Let C be the hyperbola described by the equation $1 + \frac{v^2}{b^2} - \frac{u^2}{a^2} = 0$. Define the projective transformation

$$T\colon [x : y : z] \mapsto [az : by : x].$$

For a point $\mathbf{P} = [x : y : z] \in D$, let $T(\mathbf{P}) = [x' : y' : z'] = [az : by : x]$ where

$$x = z', \quad y = \frac{y'}{b}, \quad z = \frac{x'}{a}.$$

The point $\mathbf{P} \in D$ satisfies the equation $x^2 + y^2 - z^2 = 0$ if and only if $T(\mathbf{P})$ satisfies the equation

$$0 = z'^2 + \left(\frac{y'}{b}\right)^2 - \left(\frac{x'}{a}\right)^2.$$

Restricting to the third affine chart, we see this is the equation of the hyperbola C:

$$\begin{aligned} 0 &= \frac{z'^2}{z'^2} + \frac{y'^2}{b^2 z'^2} - \frac{x'^2}{a^2 z'^2} \\ &= 1 + \frac{\left(\frac{y'}{z'}\right)^2}{b^2} - \frac{\left(\frac{x'}{z'}\right)^2}{a^2} \\ &= 1 + \frac{v^2}{b^2} - \frac{u^2}{a^2}. \end{aligned}$$

Parabola: Let C be the parabola described by the equation $v^2 - 2pu = 0$. Define the projective transformation

$$T\colon [x : y : z] \mapsto [z - x : \sqrt{2p}y : z + x].$$

For a point $\mathbf{P} = [x : y : z] \in D$, let $T(\mathbf{P}) = [x' : y' : z'] = [z - x : \sqrt{2p}y : z + x]$ where

$$x = \frac{z' - x'}{2}, \quad y = \frac{y'}{\sqrt{2p}}, \quad z = \frac{z' + x'}{2}.$$

The point $\mathbf{P} \in D$ satisfies the equation $x^2 + y^2 - z^2 = 0$ if and only if $T(\mathbf{P})$ satisfies the equation

$$\begin{aligned} 0 &= \left(\frac{z' - x'}{2}\right)^2 + \left(\frac{y'}{\sqrt{2p}}\right)^2 - \left(\frac{z' + x'}{2}\right)^2 \\ &= \frac{z'^2 - 2x'z' + x'^2}{4} + \frac{y'^2}{2p} - \frac{z'^2 + 2x'z' + x'^2}{4} \\ &= y'^2 - 2px'z'. \end{aligned}$$

Restricting to the third affine chart, we see this is the equation of the parabola C:

$$\begin{aligned} 0 &= \frac{y'^2}{z'^2} - \frac{2px'z'}{z'^2} = \left(\frac{y'}{z'}\right)^2 - 2p\left(\frac{x'}{z'}\right) \\ &= v^2 - 2pu. \end{aligned}$$

□

Corollary 5.23 *Any two conic sections are ______________________________, i.e., there exists a projective transformation from one to another.*

Proof. Projective transformations are invertible: if T is represented by the matrix M then T^{-1} is represented by the matrix M^{-1}. As a result we can map any conic section to the circle via the inverse transformation of the one constructed in Theorem 5.22. Therefore we can map any conic section to another by transforming to the circle. □

Remark 5.24 When considered as lines in $\mathbb{A}^3$, the set D is the cone shown in Figure 17. Recall that we can obtain all conic sections as intersections of this cone with a plane. It is not a coincidence that we use this same cone in the proof of Theorem 5.22. An alternative intuition behind this proof is that we are applying a linear transformation to this cone and then intersecting with the plane $z = 1$. While it is conceptually more complex, working in projective space makes this proof much easier: in $\mathbb{A}^3$ the proof is far messier!

Thank you for taking Introduction to Geometry :)