Week 10

4.2 Affine transformations of conic sections

In this section we will consider how affine transformations act on the conic sections. Initially we will prove that any ellipse can be mapped to any other ellipse via an affine transformation. The same is true for the other types of conic section: a hyperbola can be mapped to any other hyperbola; and a parabola can be mapped to any other parabola. It is not possible however, to take one type of conic section to a different type via an affine transformation. In fact, the image of an ellipse, hyperbola or parabola under an affine transformation is always again an ellipse, hyperbola or parabola respectively.

Proposition 4.16 *Let C_1 and C_2 be ellipses in $\mathbb{A}^2$. Then there is an affine transformation f such that*

$$\mathbf{P} \in C_1 \quad \textit{if and only if} \quad f(\mathbf{P}) \in C_2.$$

Proof. By Proposition 3.6 we know that there is a pair $a, b \in \mathbb{R}$ and a frame $\mathcal{F} = (\mathcal{B}, \mathbf{O})$, with $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ orthonormal such that

$$\mathbf{P} = \mathbf{O} + x\mathbf{e}_x + y\mathbf{e}_y \in C_1 \quad \text{if and only if} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Similarly there is a pair $c, d \in \mathbb{R}$, a frame $\mathcal{F} = (\mathcal{C}, \mathbf{Q})$ with $\mathcal{C} = (\mathbf{f}_x, \mathbf{f}_y)$ orthonormal such that

$$\mathbf{P} = \mathbf{Q} + x\mathbf{f}_x + y\mathbf{f}_y \in C_2 \quad \text{if and only if} \quad \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1.$$

Let $L \colon \mathbb{E}^2 \to \mathbb{E}^2$ be the linear operator given by

$$L(\mathbf{e}_1) = \frac{c}{a}\mathbf{f}_x \quad \text{and} \quad L(\mathbf{e}_2) = \frac{d}{b}\mathbf{f}_y.$$

Let f be the affine transformation defined by

$$f(\mathbf{P}) = \mathbf{Q} + L(\mathbf{P} - \mathbf{O}),$$

so

$$f(\mathbf{O} + x\mathbf{e}_x + y\mathbf{e}_y) = \mathbf{Q} + \frac{cx}{a}\mathbf{f}_x + \frac{dy}{b}\mathbf{f}_y.$$

With f as defined we have

$$\begin{aligned}
&\mathbf{P} = \mathbf{O} + x\mathbf{e}_x + y\mathbf{e}_y \in C_1 \\
&\qquad\iff \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\
&\qquad\iff \qquad \frac{\left(\frac{cx}{a}\right)^2}{c^2} + \frac{\left(\frac{dy}{b}\right)^2}{d^2} = 1 \\
&\qquad\iff \qquad f(\mathbf{P}) = \mathbf{Q} + \frac{cx}{a}\mathbf{f}_x + \frac{dy}{b}\mathbf{f}_y \in C_2.
\end{aligned}$$

□

We state the next two propositions, which concern hyperbolas and parabolas, without proofs since the arguments are very similar to those of Proposition 4.16.

Proposition 4.17 *Let C_1 and C_2 be hyperbolas in $\mathbb{A}^2$. Then there is an affine transformation f such that*

$$\mathbf{P} \in C_1 \quad \textit{if and only if} \quad f(\mathbf{P}) \in C_2.$$

Proposition 4.18 *Let C_1 and C_2 be parabolas in $\mathbb{A}^2$. Then there is an affine transformation f such that*

$$\mathbf{P} \in C_1 \quad \textit{if and only if} \quad f(\mathbf{P}) \in C_2.$$

Example 4.19 Let us consider the area of (the interior of) an ellipse, C which is given by the equation

$$\mathbf{P} = \mathbf{O} + x\mathbf{e}_x + y\mathbf{e}_y \in C \quad \text{if and only if}$$

with respect to a Cartesian frame $\mathcal{F} = \big((\mathbf{e}_x, \mathbf{e}_y), \mathbf{O}\big)$. Let U be the circle of radius 1 about $\mathbf{O}$: this has equation

$$\mathbf{P} = \mathbf{O} + x\mathbf{e}_x + y\mathbf{e}_y \in U \quad \text{if and only if}$$

Using Proposition 4.16 we have an affine transformation that maps U to C. Moreover, using the proof of the proposition, we see that the induced linear operator has matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ and determinant ab. Since the determinant controls how areas scale, and we know the area of a unit circle is π, we see that C has area $ab\pi$. Recall that a is the length of the major axis and b is the length of the minor axis.

Theorem 4.20 *Let $C \subset \mathbb{A}^2$ be a conic section and $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine transformation. Then*

- *the curve C is an ellipse if and only if $f(C)$ is an ellipse;*

- *the curve C is a hyperbola if and only if $f(C)$ is a hyperbola;*

- *the curve C is a parabola if and only if $f(C)$ is a parabola.*

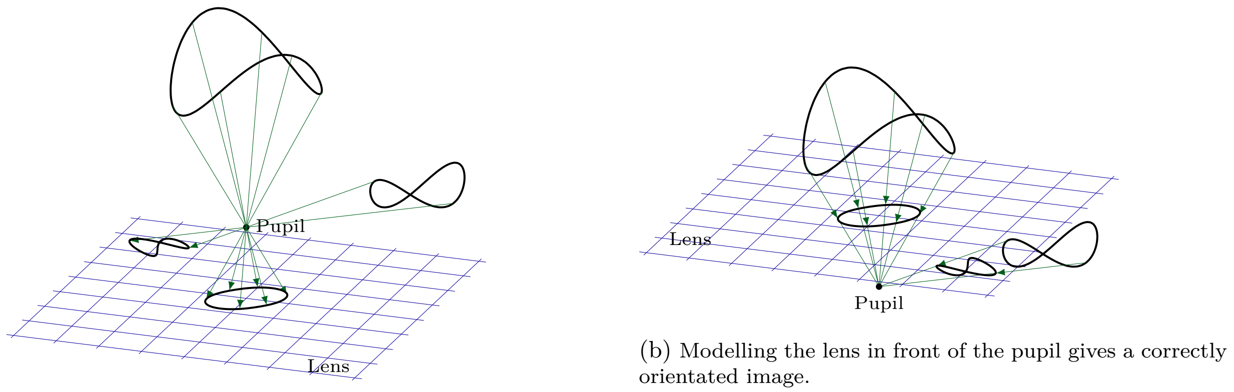
We shall not prove this theorem: the proof is most easily proved using algebraic ideas involving quadratic forms. These ideas lack much geometric intuition and hence are not particularly suited to this course. We will however make some comments on why one type of curve cannot be mapped to a different type.

Let E be an ellipse and C be a parabola or a hyperbola. The area of E is _________ (see Example 4.19) and for any affine transformation, the determinant of the associated linear operator scales this area. Since the area of C is ____________ the ellipse E cannot be mapped onto C, nor can C be mapped onto E.

To see that a hyperbola H cannot be mapped onto a parabola P, consider the two tangent lines to H that are perpendicular to the line through the foci. These lines are _____________ and each contain a single point of H: it follows that if a linear transformation mapped H to P then we would need two _____________ lines, each of which containing a single point of P, but every point on a parabola has a tangent with a different gradient, so this cannot happen.

5 Projective Geometry

Projective geometry has its origins in understanding how people see the world, this is often called perspective. Consider a human eye: light enters through the pupil and makes an image on the lens at the back of the eye. The image on the lens is upside-down and reversed but is obviously interpreted by the brain correctly. As a simplified model, let us consider the pupil as a single point and the lens a plane behind the pupil. The light hitting a particular point on the lens is determined by a straight line from that point, through the pupil. In fact, if we wish the image to be correct orientated then we can instead model the lens in front of the pupil; see Figure 18. This simple model of an eye leads to the definition of projective geometry, where the *projective points* are the lines in affine space through a fixed origin.



(a) The lens is shown behind the pupil and the image reversed.

(b) Modelling the lens in front of the pupil gives a correctly orientated image.

Figure 18: Simple model of an eye: the image on the lens is determined by straight lines from objects in space through the pupil.

Throughout this section we will implicitly use Cartesian coordinates for $\mathbb{A}^n$. This means that we assume there is a chosen origin and points will be represented as n-tuples $(x_1, \ldots, x_n)$; or often (x, y) or (x, y, z) in 2-, or 3-dimensions.

5.1 Projective space

We will begin by considering the lines in $\mathbb{A}^2$ through the origin. Let us fix the affine line $y = 1$. It straightforward to see that all lines through the origin intersect this affine line in a unique point, except the line __________, which is parallel and therefore does not intersect it at all. Since the *projective points* are the lines through the origin, we see that the points in projective space are fully determined by an affine line $\mathbb{A}^1$, together with one extra point, which we call a *point at infinity*. Since this is intuitively one-dimensional, we call this space the 1-dimensional projective space; the projective line; or $\mathbb{P}^1$. Similarly, n-dimensional projective space, or $\mathbb{P}^n$, will be the lines through the origin in $\mathbb{A}^{n+1}$.

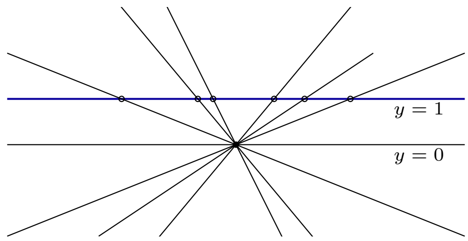


Figure 19: The projective line $\mathbb{P}^1$, is determined by points on a fixed affine line $y = 1$, together with one additional *point at infinity*.

Definition 5.1 (Projective space) We define *n-dimensional projective space*, denoted $\mathbb{P}^n$, to be the lines through the origin of the affine space

Wikipedia: n-dimensional projective space

__________.

Just as we have coordinates for $\mathbb{A}^{n+1}$, we wish to have a coordinate system for $\mathbb{P}^n$. Given a line through the origin, any point $(x_1, \ldots, x_{n+1})$ on the line, except the origin itself, determines the line. It is straightforward to see that two points $(a_1, \ldots, a_{n+1})$ and $(b_1, \ldots, b_{n+1})$ determine the same line if and only if there is a scalar λ such that $a_i = \lambda b_i$ for all $i \in \{1, \ldots, n+1\}$. Using this concept we define the *homogeneous coordinates* of projective space.

Wikipedia: homogeneous coordinate

Definition 5.2 (Homogeneous coordinates) Any $(n+1)$-tuple of real numbers, $(x_1, \ldots, x_{n+1})$, which are not ______________, determines a point in $\mathbb{P}^n$. This point is called a *homogeneous coordinate* of $\mathbb{P}^n$ and is denoted by $[x_1 : x_2 : \cdots : x_{n+1}]$. Note that for any $\lambda \neq 0$ we have

Example 5.3 Consider the projective line $\mathbb{P}^1$. Every point is given by a pair $[x : y]$, where at least one of x and y is non-zero. If $y \neq 0$ then $[x : y] =$ ____________ and these are exactly the points corresponding to lines that intersect $y = 1$ in Figure 19. On the other hand, if $y = 0$, then $x \neq 0$ and $[x : 0] =$ ____________ is the only remaining point.

Example 5.4 Generalizing Example 5.3, consider the points in $\mathbb{P}^n$. These points are given by homogeneous coordinates $[x_1 : \cdots : x_n : z]$. If $z \neq 0$ then we see that

$$[x_1 : \cdots : x_n : z] = \underline{\hspace{10em}}.$$

Since the first n values are arbitrary, we see that points of this form are in one-to-one correspondence with points in $\mathbb{A}^n$. The remaining points have $z = 0$ and have coordinates

$$[x_1 : \cdots : x_n : 0] = \underline{\hspace{12em}}$$

for any $\lambda \in \mathbb{R}$. These points are in one-to-one correspondence with points in $\mathbb{P}^{n-1}$ and are often referred to as points at infinity.

The ideas in Example 5.3 and Example 5.4 show that we can determine most points in $\mathbb{P}^n$ by points in $\mathbb{A}^n$. These ideas lead to the concept of affine charts and associated affine coordinates for projective space.

Definition 5.5 (Affine chart) Let $\mathbb{P}^n$ be a projective space with homogeneous coordinates $[x_1 : \cdots : x_{n+1}]$. To each coordinate x_i there is an associated *affine chart* $\mathbb{A}^n_i$ containing all projective points with $x_i \neq 0$:

$$\begin{aligned}\mathbb{A}^n_i &= \left\{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \;\mid\; x_i \neq 0\right\} \\ &= \left\{[u_1 : \cdots : u_{i-1} : 1 : u_i : \cdots : u_n] \in \mathbb{P}^n\right\}.\end{aligned}$$

With respect to a given chart: the points on the chart are called *finite points* and the points outside the chart are called *points at infinity*.

Wikipedia: points at infinity

In light of Definition 5.5, we see that the term *point at infinity* is a little ambiguous: a point at infinity with respect to one chart can become finite with respect to a different chart. Also notice that since a projective point cannot have all coordinates equal to zero, there always exists an affine chart containing any given projective point. That is, all projective points are finite with respect to some affine chart.

Example 5.6 Consider again the projective line $\mathbb{P}^1$ and Figure 20. The projective points that intersect the line $y = 1$ correspond to the points of the affine chart $\mathbb{A}^1_2$, where the second coordinate is non-zero. With respect to this chart the point $[1 : 0]$, corresponding to the x-axis, is a point at infinity. If instead we looked at the point of intersection with the line $x = 1$ we have the affine chart $\mathbb{A}^1_1$. The only projective point that does not intersect this line is $[0 : 1]$, corresponding to the y-axis.

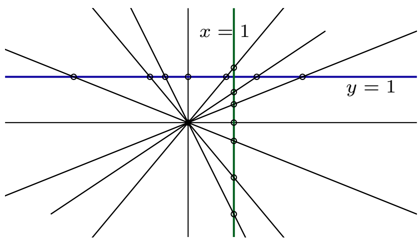


Figure 20: The projective line $\mathbb{P}^1$: the lines that intersect $y = 1$ correspond to one affine chart; the points that intersect $x = 1$ corresponds to the other.

Definition 5.7 (Affine coordinates) Let $\mathbb{A}^n_i$ be an affine chart for $\mathbb{P}^n$. Any point $\mathbf{P} = [x_1 : \cdots : x_{n+1}]$ in this chart has $x_i \neq 0$. We define the *affine coordinates* for $\mathbf{P}$ with respect to the affine chart $\mathbb{A}^n_i$ by the n-tuple $(\frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i}) = (u_1, \ldots, u_n)$, with $\frac{x_i}{x_i}$ excluded.

Wikipedia: affine coordinates

Example 5.8 Consider the point $\mathbf{P} = [1 : 2 : 0]$ in the projective plane, $\mathbb{P}^2$. What are its affine coordinates with respect to the first, second and third affine charts?

Remark 5.9 Most of the time when we fix an affine chart, it will be the last affine chart; that is the one with non-zero last coordinate.

5.2 Projective transformations

We wish to define transformations of $\mathbb{P}^n$, and we will do so using transformations of $\mathbb{A}^{n+1}$. In order for an affine transformation of $\mathbb{A}^{n+1}$ to induce a map of projective space it must take lines through the origin to lines through the origin. Recall that affine transformations of this kind are in one-to-one correspondence with invertible linear operators.

Definition 5.10 (Projective transformation) Let $T\colon \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be an affine transformation that fixes the origin (equivalently T is an invertible linear operator). The map $\mathbb{P}^n \to \mathbb{P}^n$

$$[x_1 : x_2 : \cdots : x_{n+1}] \longmapsto [x'_1 : x'_2 : \cdots : x'_{n+1}],$$

where $T(x_1, \ldots, x_{n+1}) = (x'_1, \ldots, x'_{n+1})$, is called a *projective transformation*.

Wikipedia: projective transformation

Notice that scaling T by any non-zero value gives rise to the same projective transformation.

Example 5.11 Consider the projective transformation of $\mathbb{P}^1$ given by the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. What does this map look like in terms of homogeneous and affine coordinates?

Remark 5.12 Later it will be helpful to be able to do arithmetic on the projective line. We can do arithmetic on any projective line $\mathbb{P}_1 = \mathbb{A}^1 \cup \{\infty\}$ via an abuse of notation with the following rules:

Formally, ∞ is just shorthand for the the point $[1 : 0]$ on the projective line, so you can't get away with using infinity like this unless you are working on a projective line!

In general, restricting a projective transformation to an affine chart does not give a well defined transformation. As Example 5.11 shows, these transformations tend to be defined almost everywhere on the affine chart. We call these *rational transformations*.

The formal definition of rational transformation is quite technical, therefore we won't define them properly.

When does a projective transformation define an affine transformation on an affine chart? The issue with Example 5.11 is that points on the affine chart were getting mapped to infinity and vice versa. The next proposition shows that if the projective transformation only maps affine points to affine points, it is an affine transformation. In particular, this shows that projective transformations generalize affine transformations.

Proposition 5.13 *Fix an affine chart $\mathbb{A}^n \subset \mathbb{P}^n$. Let $T\colon \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation such that for any $\mathbf{P} \in \mathbb{A}^n$ the image $T(\mathbf{P})$ is again in $\mathbb{A}^n$. Then T induces* ____________________ *of the affine chart $\mathbb{A}^n$.*

Proof. Without loss of generality we fix $\mathbb{A}^n = \mathbb{A}^n_{n+1}$, where the last coordinate is non-zero. Let

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,n} & a_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & a_{n,n+1} \\ a_{n+1,1} & \dots & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix}$$

be a matrix representation for T. (Equivalently, a matrix representation for a linear operator inducing T.) Consider the point $\mathbf{P} = [0 : \dots : 0 : 1] \in \mathbb{P}^n$. This point is in the affine chart and so $T(\mathbf{P}) = [a_{1,n+1} : \dots : a_{n+1,n+1}]$ is also in the affine chart. This means that $a_{n+1,n+1} \neq 0$ and so by rescaling the matrix we can assume the matrix is of the form

$$\begin{bmatrix} b_{1,1} & \dots & b_{1,n} & b_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ b_{n,1} & \dots & b_{n,n} & b_{n,n+1} \\ b_{n+1,1} & \dots & b_{n+1,n} & 1 \end{bmatrix} \qquad \text{where} \qquad b_{i,j} = \frac{a_{i,j}}{a_{n+1,n+1}}.$$

Now let us assume that $b_{n+1,1}$ is non-zero: then the affine chart contains the point $\left[\frac{-1}{b_{n+1,1}} : 0 : \dots : 0 : 1\right]$ and this would have an image with last coordinate 0. Since this cannot happen we conclude that $b_{n+1,1} = 0$. Similarly, we can conclude that all entries in the last row are zero apart from the last and the matrix of P has the form

$$\begin{bmatrix} b_{1,1} & \dots & b_{1,n} & b_{1,n+1} \\ \vdots & \ddots & \vdots & \vdots \\ b_{n,1} & \dots & b_{n,n} & b_{n,n+1} \\ 0 & \dots & 0 & 1 \end{bmatrix}. \tag{5.1}$$

This is exactly the matrix of an affine transformation. □

Example 5.14 Let us consider how a general projective transformation acts on an affine chart. Let $T \colon \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation given by the matrix

$$\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} \tag{5.2}$$

and with action

$$\begin{aligned} T \colon \mathbb{P}^2 &\longrightarrow \mathbb{P}^2 \\ [x : y : z] &\longmapsto [a_x x + a_y y + a_z z : b_x x + b_y y + b_z z : c_x x + c_y y + c_z z] \end{aligned}$$

What is the induced rational transformation on $\mathbb{A}^2_3$? and when is this transformation affine?