

MATH20222: Introduction to Geometry

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Introduction

Welcome to Introduction to Geometry at the University of Manchester! This semester we'll be going through a ton of various stuff in order to get you better equipped with some tools in geometry. These notes are designed to better help you learn the material and guide you through the course.

How to use these notes: These notes are different than most maths class notes. They have holes and gaps in them; they are missing information! These are the notes that we will be using in our class lectures (literally) and so it should help you follow along. The purpose of these notes is so that you don't have to write everything down; you only need to fill in the gaps giving you more time to listen to lectures and better understand the material. If you need to add extra notes, there is *tons* of room in the margins, use them! These are YOUR notes. We've also added links to Wikipedia in case you want another reference to the terms and definitions we go over. If you feel using these notes would hinder your learning experience, then *please* use another system. The point is to learn; so use whatever works best for you.

Before starting a lecture we'd recommend quickly glancing over the notes (5-10 minutes) in order to get a brief idea of what we'll be covering in class. Remember to review these notes periodically so that you don't forget terms and examples.

Good luck, we believe in you!

-Aram & David