Exercises 9

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$ and let C be the curve with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(1) Calculate the focus points for the curve C.

(2) Specify focal polar coordinates for the curve C and write down an equation for the curve in these coordinates.

Solution. (1) We have a relationship $a^2 + b^2 = c^2$ where $a^2 = 16$, $b^2 = 9$ and the focus points are at $(\pm c, 0)$. Thus $c^2 = 25$ and foci are at $(\pm 5, 0)$.

(2) Let $(-5, 0)$ be the origin for the polar coordinates and $\mathbf{e}_x$ be the $\theta = 0$ direction. Then the equation for the hyperbola is given by

$$r = \frac{p}{1 - e\cos\theta}$$

where $p = \frac{a^2-c^2}{a} = \frac{-b^2}{4} = -9/4$ and $e = \frac{c}{a} = 5/4$. Thus the equation of C in polar coordinates is

$$r = \frac{-9}{4 - 5\cos\theta}.$$

□

Question 2 Fix a Cartesian coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ for $\mathbb{A}^n$. Let $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ and $g(\mathbf{x}) = G(\mathbf{x}) + \mathbf{Q}$ be two affine transformations. [So F and G are invertible linear maps.]

(1) Prove that the composition $h := f \circ g$ is also an affine transformation $h(\mathbf{x}) = H(\mathbf{x}) + \mathbf{R}$. *[Hint: $h(\mathbf{x}) = f(g(\mathbf{x}))$.]*

(2) Let h denote the identity transformation $h(\mathbf{x}) = \mathbf{x}$. Suppose $h = f \circ g$ is the identity. What are G and $\mathbf{Q}$ in terms of F and $\mathbf{P}$?

[In particular, the above shows that the set of affine transformations is a *group*, denoted AGL_n.]

Solution. (i) $f(G(\mathbf{x}) + \mathbf{Q}) = (F \circ G)(\mathbf{x}) + F(\mathbf{Q}) + \mathbf{P}$. So $H = F \circ G$ and $\mathbf{R} = F(\mathbf{Q}) + \mathbf{P}$. Since F and G are invertible, H is invertible.

(ii) If $H = id$ and $\mathbf{P} = \mathbf{O}$ then $G = F^{-1}$ and $\mathbf{Q} = -F^{-1}(\mathbf{P})$. □

Question 3 Let f be the affine map $f : \mathbb{A}^2 \to \mathbb{A}^2$ that is a reflection in the line $x = 1$. Suppose $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ for a linear invertible map F and vector $\mathbf{P} \in \mathbb{A}^2$.

(1) Determine the vector $\mathbf{P}$.

(2) Work out where f sends the points $(1, 0)$ and $(0, 1)$. Hence, write down the matrix of $F\colon \mathbb{E}^2 \to \mathbb{E}^2$ with respect to the standard basis on $\mathbb{A}^2$.

Solution. (1) The point $(0, 0)$ is mapped to $(2, 0)$ by f so

$$f(\mathbf{O}) = \mathbf{P} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(2) $f(1, 0) = (1, 0)$, so $F(1, 0) = (1, 0) - (2, 0) = (-1, 0)$. $f(0, 1) = (2, 1)$, so $F(0, 1) = (2, 1) - (2, 0) = (0, 1)$. Therefore

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Question 4 Consider the affine transformation $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ on $\mathbb{A}^2$ that is made up of a rotation F by $\frac{\pi}{3}$ about the origin, followed by a translation by $\mathbf{P} = (3, 2)$.

(1) Write down the matrix of the linear operator F with respect to the standard basis.

(2) Find a point fixed by f.

(3) Describe the affine transformation f in a coordinate system where the fixed point is taken to be the origin.

Solution. (1)

$$F = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(2)

$$\begin{aligned} f(x,y) = (x,y) &\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x - \frac{\sqrt{3}}{2}y + 3 \\ \frac{\sqrt{3}}{2}x + \frac{1}{2}y + 2 \end{bmatrix} \\ &\iff \begin{cases} \frac{1}{2}x = 3 - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x = \frac{1}{2}y - 2 \end{cases} \\ &\iff \begin{cases} \frac{1}{2}x = 3 - \frac{\sqrt{3}}{2}y \\ \sqrt{3}(3 - \frac{\sqrt{3}}{2}y) = \frac{1}{2}y - 2 \end{cases} \\ &\iff y = 1 + \frac{3\sqrt{3}}{2} \text{ and } x = \frac{3}{2} - \sqrt{3} \end{aligned}$$

(3) f is a rotation of $\pi/3$ about the fixed point.

□

Question 5 Let $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ be the affine transformation with

$$F = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}$$

(1) Find a line in $\mathbb{A}^2$ that is fixed by f.

(2) Show that the induced linear operator F of f is orthogonal.

(3) Deduce that f is a reflection in the fixed line.

Solution. (1)

$$\begin{aligned} f(x,y) = (x,y) &\iff F(x,y) + \mathbf{P} = (x,y) \\ &\iff \begin{cases} 3x - 4y + 6 = 5x \\ -4x - 3y + 12 = 5y \end{cases} \\ &\iff \begin{cases} 2x + 4y = 6 \\ 4x + 8y = 12 \end{cases} \\ &\iff x + 2y = 3 \end{aligned}$$

The line given by the equation $x + 2y = 3$ is fixed by f.

(2) Check that $FF^T = Id$.

(3) The linear operator has determinant given by

$$\det \begin{bmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{bmatrix} = -9/25 - 16/25 = -1,$$

which is negative and since it is orthogonal acting on $\mathbb{E}^2$ it is a reflection.

□