Exercises 8

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following curves C and 1-forms ω, calculate $\int_C \omega$ up to sign.

(1) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid y = x^2,\, 0 < x < 1 \right\}, \quad \omega = y\mathbf{d}x - x\mathrm{d}y$

(2) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid y^3 = x,\, -1 < x < 1 \right\}, \quad \omega = xy\mathbf{d}x + xy\mathbf{d}y$

(3) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid x^2 + y^2 = 13 \right\}, \quad \omega = (2xy + y^2)\mathbf{d}x + (x^2 + 2xy)\mathbf{d}y$

(4) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid x^2 - y^2 = 1,\, 1 < x < 2,\, y > 0 \right\}, \quad \omega = e^x\mathbf{d}x + \mathbf{d}y$

Solution. For all solution, we compute the integral for a specific parametrisation. Your answer may differ by sign if you picked a parametrisation with opposite orientation.

(1) One parametrisation for C is $\boldsymbol{\gamma}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ on the interval $(0, 1)$ with velocity vector $\boldsymbol{\gamma}'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$. Then the integral takes the value

$$\int_C y\mathbf{d}x - x\mathrm{d}y = \int_0^1 t^2 \cdot 1 - t \cdot 2t\mathbf{d}t = \int_0^1 -t^2\mathbf{d}t = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3}.$$

(2) One parametrisation for C is $\boldsymbol{\gamma}(t) = \begin{pmatrix} t^3 \\ t \end{pmatrix}$ on the interval $(-1, 1)$ with velocity vector $\boldsymbol{\gamma}'(t) = \begin{bmatrix} 3t^2 \\ 1 \end{bmatrix}$. Then the integral takes the value

$$\int_C xy\mathrm{d}x + xy\mathrm{d}y = \int_{-1}^1 t^4 \cdot 3t^2 + t^4 \cdot 1\mathbf{d}t = \int_{-1}^1 3t^6 + t^4\mathbf{d}t = \left[\frac{3t^7}{7} + \frac{t^5}{5} \right]_{-1}^1 = \frac{44}{35}$$

(3) ω is exact, as it is of the form $\omega = \mathbf{d}(x^2y + xy^2)$. Furthermore, C is a closed curve and so $\int_C \omega = 0$.

(4) ω is exact as $\omega = \mathbf{d}f$ where $f = e^x + y$. Therefore we just need to evaluate f on the endpoints of C. The endpoints are $(2, \sqrt{3}), (1, 0)$ and so

$$\int_C \omega = f(2, \sqrt{3}) - f(1, 0) = \left(e^2 + \sqrt{3}\right) - \left(e^1 + \sqrt{0}\right) = e(e-1) + \sqrt{3}.$$

□

Question 2 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be an ellipse with foci $\mathbf{F}_1 = (-1, 0)$ and $\mathbf{F}_2 = (1, 0)$. The point $\mathbf{P} = (0, 2)$ is on the ellipse C.

(1) Find the implicit equation for the ellipse in Cartesian coordinates.

(2) What is the eccentricity, e, of the ellipse.

Solution. (1) The distance of the ellipse is $2a$ where

$$\begin{aligned} 2a &= \|\mathbf{F}_1 - \mathbf{P}\| + \|\mathbf{F}_2 - \mathbf{P}\| \\ &= \sqrt{2^2 + 1^2} + \sqrt{2^2 + 1^2} \\ &= 2\sqrt{5}. \end{aligned}$$

Now $b^2 = a^2 - c^2 = 5 - 1 = 4$ so $b = 2$. Alternatively we can note that we have the standard coordinate system for the ellipse so the point $(0, b) = (0, 2)$ is on the curve. Thus the equation for the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{4} = 1.$$

(2) The eccentricity is $e = \frac{c}{a} = \frac{1}{\sqrt{5}}$.

□

Question 3 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be a hyperbola with focus points $\mathbf{F}_1 = (-2, 0)$ and $\mathbf{F}_2 = (2, 0)$ and assume the point $\mathbf{P} = (2, 3)$ is on the curve C.

(1) Find the two points $(s, 0)$ and $(t, 0)$ where C intersects the x-axis.

(2) Write down the implicit equation of the curve.

Solution. We must calculate the distance $2a$ of the hyperbola:

$$\begin{aligned} 2a &= |\|\mathbf{F}_1 - \mathbf{P}\| - \|\mathbf{F}_2 - \mathbf{P}\|| \\ &= \left|\sqrt{4^2 + 3^2} - \sqrt{3^2}\right| \\ &= 2 \end{aligned}$$

(1) Since $a = 1$ and we have the standard coordinate system the points of intersection with the x-axis are at $(-1, 0)$ and $(1, 0)$.

(2) $b^2 = c^2 - a^2 = 2^2 - 1^2 = 3$ so the equation for the hyperbola is

$$x^2 - \frac{y^2}{3} = 1.$$

□

2 Extra Exercises

Question 4 Consider the curve

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x = 2y^2 - 1,\ 0 < y < 1 \right\}.$$

(1) Compute two parametrisations of C with the opposite orientation.

(2) Calculate the integral

$$\int_C \sin y \mathbf{d}x$$

for each parametrisation.

Solution. (1) By letting $y = t$, we get the parametrisation of C

$$\begin{aligned} \boldsymbol{\gamma} \colon (0,1) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} 2t^2 - 1 \\ t \end{pmatrix} \end{aligned}$$

The reparametrisation map

$$\varphi \colon (-1, 0) \to (0, 1), \quad \varphi(t) = -t$$

allows us to define another parametrisation $\widetilde{\boldsymbol{\gamma}}(t)$ of C:

$$\begin{aligned} \widetilde{\boldsymbol{\gamma}} \colon (-1,0) &\to \mathbb{A}^2 \\ t &\mapsto \boldsymbol{\gamma}(\varphi(t)) = \begin{pmatrix} 2t^2 - 1 \\ -t \end{pmatrix} \end{aligned}$$

As $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -1$, $\widetilde{\boldsymbol{\gamma}}$ has the opposite orientation to $\boldsymbol{\gamma}$.

Note: There are loads of different parametrisations you could obtain for this curve. However, the following part requires quite a complicated integral, therefore the more simple you can make your parametrisation, the better.

(2) Note that $\sin y \mathbf{d}x$ is not exact, therefore we must use a parametrisation to evaluate the integral. The corresponding velocity vectors for our parametrisations calculated in part (a) are

$$\boldsymbol{\gamma}'(t) = \begin{bmatrix} 4t \\ 1 \end{bmatrix}, \quad \widetilde{\boldsymbol{\gamma}}'(t) = \begin{bmatrix} 4t \\ -1 \end{bmatrix}$$

For $\boldsymbol{\gamma}$, we have the integral

$$\begin{aligned} \int_C \sin y \mathbf{d}x &= \int_0^1 4t \cdot \sin t \mathbf{d}t \\ &= [-4t \cos t]_0^1 + \int_0^1 4 \cos t \mathbf{d}t \quad \text{(integration by parts)} \\ &= [-4t \cos t + 4 \sin t]_0^1 \\ &= 4 \sin 1 - 4 \cos 1 \end{aligned}$$

For $\widetilde{\boldsymbol{\gamma}}$, the calculation is very similar:

$$\begin{aligned} \int_C \sin y \mathbf{d}x &= \int_{-1}^0 4t \cdot \sin(-t) \mathbf{d}t \\ &= -\int_{-1}^0 4t \cdot \sin t \mathbf{d}t \\ &= -\left[-4t \cos t + 4 \sin t\right]_{-1}^0 \quad \text{(by previous integral)} \\ &= 4 \cos(-1) + 4 \sin(-1) \\ &= -(4 \sin(1) - 4 \cos(1)) \end{aligned}$$

This is the solution we expect, as we simply reversed the orientation of the curve.

□

Question 5 Calculate the integral

$$\int_C x\mathbf{d}y$$

for each of the following curves:

(1) the circle $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 + y^2 = 12y \right\}$

(2) the ellipse $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 + \frac{y^2}{9} = 1 \right\}$.

Solution. $x\mathrm{d}y$ is not exact, therefore we must calculate parametrisations for both of these curves.

(1) By completing the square, we see C has the equation $x^2 + (y-6)^2 = 36$, *i.e.*, it is the circle of radius 6 with centre $(0, 6)$. We can modify the standard parametrisation of a circle to get

$$\begin{aligned} \boldsymbol{\gamma}\colon [0, 2\pi] &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} 6\cos t \\ 6\sin t + 6 \end{pmatrix}, \quad \boldsymbol{\gamma}(t) = \begin{bmatrix} -6\sin t \\ 6\cos t \end{bmatrix}. \end{aligned}$$

Using this parametrisation, we obtain the integral

$$\begin{aligned} \int_C x\mathrm{d}y &= \int_0^{2\pi} 36\cos^2 t\mathbf{d}t \\ &= \int_0^{2\pi} 18\cos 2t + 18 \quad \left(\cos^2 t = \frac{1}{2}(\cos 2t + 1)\right) \\ &= [9\sin 2t + 18t]_0^{2\pi} \\ &= 36\pi. \end{aligned}$$

(2) We can modify the standard parametrisation of a circle to get

$$\begin{aligned} \boldsymbol{\gamma}\colon [0, 2\pi] &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} \cos t \\ 3\sin t \end{pmatrix}, \quad \boldsymbol{\gamma}(t) = \begin{bmatrix} -\sin t \\ 3\cos t \end{bmatrix}. \end{aligned}$$

Using this parametrisation, we obtain the integral

$$\begin{aligned} \int_C x\mathrm{d}y &= \int_0^{2\pi} 3\cos^2 t\mathbf{d}t \\ &= \int_0^{2\pi} \frac{3}{2}\cos 2t + \frac{3}{2} \quad \left(cos^2 t = \frac{1}{2}(\cos 2t + 1)\right) \\ &= \left[\frac{3}{4}\sin 2t + \frac{3}{2}t\right]_0^{2\pi} \\ &= 3\pi. \end{aligned}$$

□

Question 6 Let C be the circle with radius a centred at the origin. Recall that we can parametrise C via

$$\begin{aligned} \boldsymbol{\gamma}\colon [0, 2\pi) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}. \end{aligned}$$

(1) Write $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ in polar coordinates (r, θ).

(2) Compute the integral

$$\int_C r^2\mathbf{d}\theta.$$

(3) How does the integral in part (b) relate to the following integral?

$$\int_C -y\mathbf{d}x + x\mathbf{d}y$$

Solution. (1) As $x = a\cos t$ and $y = a\sin t$, it is immediate that $r = a$ and $\theta = t$ in polar coordinates. We can therefore parametrise C in polar coordinates as

$$\begin{aligned}\boldsymbol{\gamma}\colon [0, 2\pi) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} a \\ t \end{pmatrix}.\end{aligned}$$

(2) We calculate this integral the same way we do in Cartesian coordinates. The velocity vector of $\boldsymbol{\gamma}$ in polar coordinates is $\boldsymbol{\gamma}'(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Substituting this into the integral gives

$$\int_C r^2 \mathbf{d}\theta = \int_0^{2\pi} a^2 \cdot 1 = 2\pi a^2.$$

(3) We showed on sheet 6 that $-y\mathbf{d}x + x\mathbf{d}y$ and $r^2\mathbf{d}\theta$ are the same 1-form in different coordinate systems. As the value of $\int_C \omega$ is invariant under changing coordinates, we can immediate deduce that

$$\int_C -y\mathbf{d}x + x\mathbf{d}y = 2\pi a^2.$$

We can explicitly calculate the value of $\int_C -y\mathbf{d}x + x\mathbf{d}y$ *to verify this. Using the parametrisation* $\boldsymbol{\gamma}$*, we get*

$$\begin{aligned}\int_C -y\mathbf{d}x + x\mathbf{d}y &= \int_0^{2\pi} -a\sin t \cdot -a\sin t + a\cos t \cdot a\cos t\mathbf{d}t \\ &= \int_0^{2\pi} a^2(\sin^2 t + \cos^2 t)\mathbf{d}t = 2\pi a^2.\end{aligned}$$

□

Question 7 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be an ellipse with focus points $\mathbf{F}_1 = (-5, 0)$ and $\mathbf{F}_2 = (16, 0)$. The point $\mathbf{P} = (0, 12)$ is on the ellipse.

(1) Find the two points of intersection between C and the line $x = 0$ (that is, the y-axis).

(2) Find the two points of intersection between C and the line $y = 0$ (that is, the x-axis).

(3) What is the eccentricity of the ellipse?

Solution. First notice that the focal line is the x-axis, but we do not have the standard coordinates for the ellipse.

(1) Since the focal line is the x-axis, we have symmetry in this line. Therefore $(0, 12)$ and $(0, -12)$ are on the curve.

(2) We need to know the distance of the curve:

$$\begin{aligned}2a &= \|\mathbf{F}_1 - \mathbf{P}\| + \|\mathbf{F}_2 - \mathbf{P}\| \\ &= \sqrt{5^2 + 12^2} + \sqrt{16^2 + 12^2} \\ &= \sqrt{169} + 4\sqrt{4^2 + 3^2} = 13 + 20 = 33\end{aligned}$$

The centre of the ellipse is at $({}^{21}/_2 - 5, 0) = ({}^{11}/_2, 0)$, so the ellipse intersects the focal line $y = 0$ at the points $(\frac{11-33}{2}, 0) = (-11, 0)$ and $(\frac{11+33}{2}, 0) = (22, 0)$.

(3) $e = \frac{c}{a} = \frac{21/2}{33/2} = \frac{7}{11}$.

□

Question 8 Fix Cartesian coordinates (x, y) and let C_1, C_2 and C_3 be conic sections with the implicit equations:

$$C_1\colon 4x^2 + 4x + y^2 = 0;$$
$$C_2\colon 4x^2 + 4x - y^2 = 0;$$
$$C_3\colon 4x^2 + 4x + y = 0.$$

Show that C_1 is an ellipse, C_2 is a hyperbola and C_3 is a parabola.
Solution. Starting with C_1 we see

$$\begin{aligned} & & 4x^2 + 4x + y^2 &= 0 \\ &\iff & 4x^2 + 4x + 1 + y^2 &= 1 \\ &\iff & 4(x^2 + x + 1/4) + y^2 &= 1 \\ &\iff & 4(x + 1/2)^2 + y^2 &= 1 \end{aligned}$$

So with Cartesian coordinates $(\tilde{x}, y)$ where $\tilde{x} = x + 1/2$, the equation of C_1 is $4\tilde{x}^2 + y^2 = 1$: the standard equation for an ellipse. These new coordinates have the same orthonormal basis as the original Cartesian coordinate system, but the origin is shifted to $(x, y) = (-1/2, 0)$.

Similarly for C_2 we see that

$$4x^2 + 4x - y^2 = 0 \qquad \iff \qquad 4(x + 1/2)^2 - y^2 = 1$$

is the standard equation for a hyperbola in the coordinates $(\tilde{x}, y)$.

Finally

$$\begin{aligned} & & 4x^2 + 4x + y &= 0 \\ &\iff & 4x^2 + 4x + 1 + y - 1 &= 0 \\ &\iff & 4x^2 + 4x + 1 + y - 1 &= 0 \\ &\iff & 1 - y &= 4(x + 1/2)^2 \end{aligned}$$

Therefore using coordinates $(\tilde{x}, \tilde{y})$ where $\tilde{x}$ is as above and $\tilde{y} = 1 - y$ the equation for C_3 is $\tilde{y} = 4\tilde{x}^2$: a standard equation for a parabola. These new coordinates have a different orthonormal basis and origin from the original coordinate system. The new orthonormal basis is $(\mathbf{e}_x, -\mathbf{e}_y)$, where $(\mathbf{e}_x, \mathbf{e}_y)$ is the orthonormal basis for the original Cartesian coordinates. The new origin is at $(x, y) = (-1/2, 1)$. □