Exercises 7

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following parametrisations $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$

(1) show $\boldsymbol{\gamma}_1$ is a reparametrisation of $\boldsymbol{\gamma}_2$,

(2) state whether they have the same orientation.

$$\boldsymbol{\gamma}_1 \colon (0,1) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix} \qquad \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{4}\right) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} \tag{1}$$

$$\boldsymbol{\gamma}_1 \colon (0,1) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} t \\ 2t^2 - 1 \end{pmatrix} \qquad \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{2}\right) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} \cos t \\ \cos 2t \end{pmatrix} \tag{2}$$

$$\boldsymbol{\gamma}_1 \colon (0,1) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} 1-t \\ 1-2t \end{pmatrix} \qquad \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{2}\right) \to \mathbb{A}^2, \quad t \mapsto \begin{pmatrix} \sin^2 t \\ -\cos 2t \end{pmatrix} \tag{3}$$

Solution. For each pair of parametrisations, we find a map $\varphi \colon I_2 \to I_1$ and show that it satisfies

- $\boldsymbol{\gamma}_1(\varphi(t)) = \boldsymbol{\gamma}_2(t) \quad \forall t \in I_2$,
- $\frac{\mathrm{d}\varphi}{\mathrm{d}t} \neq 0 \quad \forall t \in I_2$.

Note that we shall we shall usually skip the proof that φ is smooth and bijective, as it is usually clear, and these are standard algebraic checks.

Parametrisations (1):

$$\varphi \colon \left(0, \frac{\pi}{4}\right) \to (0,1), \quad t \mapsto \sin 2t$$

$$\boldsymbol{\gamma}_1(\varphi(t)) = \boldsymbol{\gamma}_1(\sin 2t) = \begin{pmatrix} \sin 2t \\ \sqrt{1 - \sin^2 2t} \end{pmatrix} = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} = \boldsymbol{\gamma}_2(t)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 2\cos 2t > 0 \quad \forall t \in \left(0, \frac{\pi}{4}\right)$$

Therefore φ is a reparametrisation map. As $\frac{\mathrm{d}\varphi}{\mathrm{d}t} > 0$, these parametrisations have the same orientation.

Parametrisations (2):

$$\varphi \colon \left(0, \frac{\pi}{2}\right) \to (0,1), \quad t \mapsto \cos t$$

$$\boldsymbol{\gamma}_1(\varphi(t)) = \boldsymbol{\gamma}_1(\cos t) = \begin{pmatrix} \cos t \\ 2\cos^2 t - 1 \end{pmatrix} = \begin{pmatrix} \cos t \\ \cos 2t \end{pmatrix} = \boldsymbol{\gamma}_2(t)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\sin t < 0 \quad \forall t \in \left(0, \frac{\pi}{2}\right)$$

Therefore φ is a reparametrisation map. As $\frac{\mathrm{d}\varphi}{\mathrm{d}t} < 0$, these parametrisations have the opposite orientation.

Parametrisations (3):

$$\varphi \colon \left(0, \frac{\pi}{2}\right) \to (0,1), \quad t \mapsto \cos^2 t$$

$$\gamma_1(\varphi(t)) = \gamma_1(\cos^2 t) = \begin{pmatrix} 1-\cos^2 t \\ 1-2\cos^2 t \end{pmatrix} = \begin{pmatrix} \sin^2 t \\ -\cos 2t \end{pmatrix} = \gamma_2(t)$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -2\sin\cos t = -\sin 2t < 0 \quad \forall t \in \left(0, \frac{\pi}{2}\right)$$

Therefore φ is a reparametrisation map. As $\frac{\mathrm{d}\varphi}{\mathrm{d}t} < 0$, these parametrisations have the opposite orientation. □

Question 2 Consider the curve C

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \;\middle|\; y^2 = x^3+1,\, -1 \le x \le 1,\, y > 0 \right\},$$

and the point on the curve $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(1) By calculating $\frac{\mathrm{d}y}{\mathrm{d}x}$, make a prediction what the tangent space $T_{\mathbf{P}}(C)$ to the curve at $\mathbf{P}$ will be.

(2) For each of the following vectors, find a parametrisation γ of C such that it is the velocity vector γ' at $\mathbf{P}$:

$$\text{(i)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution. (1) As $y = +\sqrt{x^3+1}$, we calculate $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\sqrt{x^3+1}}$. Intuitively, we know the derivative evaluated at the point $\mathbf{P}$ as the slope of the tangent line at the point $\mathbf{P}$. Evaluating, we get $\frac{\mathrm{d}y}{\mathrm{d}x}(\mathbf{P}) = 0$ and so we expect the tangent line at $\mathbf{P}$ to be $y = 1$, or more formally that the tangent space is $T_{\mathbf{P}}(C) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Note that we did not formally prove this would be the case, it is just intuition. Part (b) will show our intuition to be correct.

(2) Note that these are not the only parametrisations we may pick, there are many many more.

(i) Setting $x = t$, we get the following parametrisation and velocity vector:

$$\gamma\colon [-1,1] \to \mathbb{A}^2$$
$$t \mapsto \begin{pmatrix} t \\ \sqrt{t^3+1} \end{pmatrix} \qquad \gamma'(t) = \begin{bmatrix} 1 \\ \frac{3t^2}{2\sqrt{t^3+1}} \end{bmatrix}$$

The parametrisation hits the point $\mathbf{P}$ at time $t = 0$, therefore the velocity vector at $\mathbf{P}$ is $\gamma'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(ii) Setting $x = -3t$, we get the following parametrisation and velocity vector:

$$\gamma\colon \left[-\frac{1}{3}, \frac{1}{3}\right] \to \mathbb{A}^2$$
$$t \mapsto \begin{pmatrix} -3t \\ \sqrt{1-27t^3} \end{pmatrix} \qquad \gamma'(t) = \begin{bmatrix} -3 \\ \frac{-81t^2}{2\sqrt{1-27t^3}} \end{bmatrix}$$

The parametrisation hits the point $\mathbf{P}$ at time $t = 0$, therefore the velocity vector at $\mathbf{P}$ is $\gamma'(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

(iii) We would like both entries of $\gamma'(t)$ to have a dependency on t, so setting $t = 0$ will give the zero velocity vector. Setting $x = t^3$, we get the following parametrisation and velocity vector:

$$\gamma\colon [-1,1] \to \mathbb{A}^2$$
$$t \mapsto \begin{pmatrix} t^3 \\ \sqrt{t^9+1} \end{pmatrix} \qquad \gamma'(t) = \begin{bmatrix} 3t^2 \\ \frac{9t^8}{2\sqrt{t^9+1}} \end{bmatrix}$$

The parametrisation hits the point $\mathbf{P}$ at time $t = 0$, therefore the velocity vector at $\mathbf{P}$ is $\gamma'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(Note: you must be careful picking this parametrisation that we hit all the points in C. For example, if our parametrisation sets $x = t^2$, then $\boldsymbol{\gamma}(t) = (t^2, \sqrt{t^6 + 1})$ will not hit the point $(-1, 0)$ as no value of t can realise $t^2 = -1$.)

□

2 Extra Exercises

Question 3 Consider a 0-form $f(x,y)$ on $\mathbb{A}^2$. In terms of partial derivatives of f:

(1) find a vector field $\mathbf{W}$ such that $D_{\mathbf{W}}f$ is zero for all points in $\mathbb{A}^2$,

(2) find a parametrisation $\boldsymbol{\gamma}$ of a curve C such that $D_{\boldsymbol{\gamma}'}f$ is zero for all points in C.

Solution. (1) Let $\mathbf{W} = h_x\mathbf{e}_x + h_y\mathbf{e}_y$ where h_x, h_y are real-valued functions on $\mathbb{A}^2$. By the definition of directional derivatives, we have

$$D_{\mathbf{W}}f = \frac{\partial f}{\partial x}h_x + \frac{\partial f}{\partial y}h_y.$$

If $D_{\mathbf{W}}f = 0$, then we need to find functions h_x, h_y that satisfy

$$\frac{\partial f}{\partial x}h_x + \frac{\partial f}{\partial y}h_y = 0.$$

The functions $h_x = -\frac{\partial f}{\partial y}, h_y = \frac{\partial f}{\partial x}$ will satisfy, and so $D_{\mathbf{W}}f$ is zero on the vector field $\mathbf{W} = -\frac{\partial f}{\partial y}\mathbf{e}_x + \frac{\partial f}{\partial x}\mathbf{e}_y$.

(2) Let $\boldsymbol{\gamma}(t)$ be a parametrisation with velocity vector $\boldsymbol{\gamma}'(t)$ given as

$$\boldsymbol{\gamma}(t) = \begin{pmatrix}\gamma_x(t)\\ \gamma_y(t)\end{pmatrix}, \quad \boldsymbol{\gamma}'(t) = \begin{bmatrix}\gamma_x'(t)\\ \gamma_y'(t)\end{bmatrix} = \begin{bmatrix}\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}\\ \frac{\mathrm{d}\gamma_y}{\mathrm{d}t}\end{bmatrix}.$$

By the definition of directional derivative along a curve, we have

$$D_{\boldsymbol{\gamma}'}f = \frac{\partial f}{\partial x}\gamma_x' + \frac{\partial f}{\partial y}\gamma_y'$$

If $D_{\boldsymbol{\gamma}'}f = 0$, as before we can set

$$\gamma_x'(t) = -\frac{\partial f}{\partial y}(\boldsymbol{\gamma}(t)), \quad \gamma_y'(t) = \frac{\partial f}{\partial x}(\boldsymbol{\gamma}(t))$$

Therefore $\boldsymbol{\gamma}(t)$ is a solution to the above differential equation. *(Given that $\boldsymbol{\gamma}$ is in such a general form, we cannot do better than stating it as a solution to a PDE rather than as an explicit formula.)*

Remark: As $\boldsymbol{\gamma}'$ is a "sub-vector field" of $\mathbf{W}$, we can think of $\boldsymbol{\gamma}$ as a point moving along $\mathbf{W}$ from a starting position. This will always trace out some curve, and so will always exist. However, if f is particularly horrible, then the curve may also be quite horrible to express.

□

Question 4 For each of the following implicit curves, find two parametrisations whose orientations are opposite.

(1) $C = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{A}^2 \mid \frac{x^2}{3} + \frac{y^2}{5} = 1\right\}$

(2) $C = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{A}^2 \mid x^2 - y^2 = 1\right\}$,

Solution. (1) You may recognise C as an ellipse. As a result, we can parametrise it in a similar way to the circle:

$$\begin{aligned}\boldsymbol{\gamma}\colon [0,2\pi] &\to \mathbb{A}^2\\ t &\mapsto \begin{pmatrix}\sqrt{3}\cos t\\ \sqrt{5}\sin t\end{pmatrix}\end{aligned}$$

To find a parametrisation with an opposite orientation, we can use a reparametrisation map φ whose derivative is negative. The easiest way to do this is via the map

$$\begin{aligned}\varphi\colon [0,2\pi] &\to [0,2\pi]\\ t &\mapsto 2\pi - t\end{aligned}$$

We use φ to reparametrise $\boldsymbol{\gamma}$:

$$\begin{aligned}\boldsymbol{\gamma}'\colon [0, 2\pi] &\to \mathbb{A}^2 \\ t &\mapsto \boldsymbol{\gamma}(\varphi(t)) = \begin{pmatrix}\sqrt{3}\cos(2\pi - t) \\ \sqrt{5}\sin(2\pi - t)\end{pmatrix}\end{aligned}$$

As φ is a reparametrisation map with $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -1 < 0$, these two parametrisations have opposite orientation.

(2) You may recognise C as a hyperbola. Using the trig identity $\sec^2 t - \tan^2 t = 1$, we can parametrise C as

$$\begin{aligned}\boldsymbol{\gamma}\colon [0, 2\pi] &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix}\sec t \\ \tan t\end{pmatrix}\end{aligned}$$

Finding a parametrisation with the opposite orientation can be done via precisely the same reparametrisation map φ from part (a):

$$\begin{aligned}\boldsymbol{\gamma}'\colon [0, 2\pi] &\to \mathbb{A}^2 \\ t &\mapsto \boldsymbol{\gamma}(\varphi(t)) = \begin{pmatrix}\sec(2\pi - t) \\ \tan(2\pi - t)\end{pmatrix}\end{aligned}$$

□