Exercises 6

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Plot each of the following vector fields in $\mathbb{A}^2$. Determine whether they are conservative or not.

(1) $\mathbf{W} = -x\mathbf{e}_x - y\mathbf{e}_y$

(2) $\mathbf{W} = y\mathbf{e}_x + x\mathbf{e}_y$

(3) $\mathbf{W} = (x+y)\mathbf{e}_x + (x+y)\mathbf{e}_y$

Solution. The plots of the three vector fields are given in Figure 1.

(1) $\mathbf{W}$ is conservative: for the function $f(x,y) = \frac{-x^2-y^2}{2}$, we have

$$\frac{\partial f}{\partial x} = -x, \frac{\partial f}{\partial y} = -y \Rightarrow \nabla f = -x\mathbf{e}_x - y\mathbf{e}_y.$$

(2) $\mathbf{W}$ is conservative: for the function $f(x,y) = xy$, we have

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x \Rightarrow \nabla f = y\mathbf{e}_x + x\mathbf{e}_y.$$

(3) $\mathbf{W}$ is conservative: for the function $f(x,y) = \frac{(x+y)^2}{2}$, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = x+y \Rightarrow \nabla f = (x+y)\mathbf{e}_x + (x+y)\mathbf{e}_y.$$

□

Question 2 For each of the following 1-forms ω, find all points $\mathbf{P} \in \mathbb{A}^2$ such that $\omega_{\mathbf{P}} := \omega(\mathbf{P}, -)$ is the linear functional

$$\begin{aligned} \omega_{\mathbf{P}} \colon T_{\mathbf{P}}\left(\mathbb{A}^2\right) &\to \mathbb{R} \\ \begin{bmatrix} v_x \\ v_y \end{bmatrix} &\mapsto v_x + v_y. \end{aligned}$$

(1) $\omega = x\mathbf{d}x + y\mathbf{d}y$

(2) $\omega = 3x\mathbf{d}x + y^2\mathbf{d}y$

(3) $\omega = \sin x\mathbf{d}x + \cos y\mathbf{d}y$

Solution. Recall that for a 1-form $\omega = g_x\mathbf{d}x + g_y\mathbf{d}y$, the linear functional $\omega_{\mathbf{P}}$ is

$$\omega_P(\mathbf{v}) = g_x(\mathbf{P})\mathbf{d}x(\mathbf{v}) + g_y(\mathbf{P})\mathbf{d}y(\mathbf{v}) = g_x(\mathbf{P})v_x + g_y(\mathbf{P})v_y.$$

Therefore for each ω, we want the points $\mathbf{P}$ such that $g_x(\mathbf{P}) = g_y(\mathbf{P}) = 1$.

(1) If $x = y = 1$, the unique point with this linear functional is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(2) If $3x = y^2 = 1$, then there are two points with this linear functional

$$\left\{ \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -1 \end{pmatrix} \right\}.$$

(3) If $\sin x = \cos y = 1$, then the set of points with this linear functional are

$$\left\{ \begin{pmatrix} \frac{\pi}{2} + 2m\pi \\ 2n\pi \end{pmatrix} \;\middle|\; m, n \in \mathbb{Z} \right\}.$$

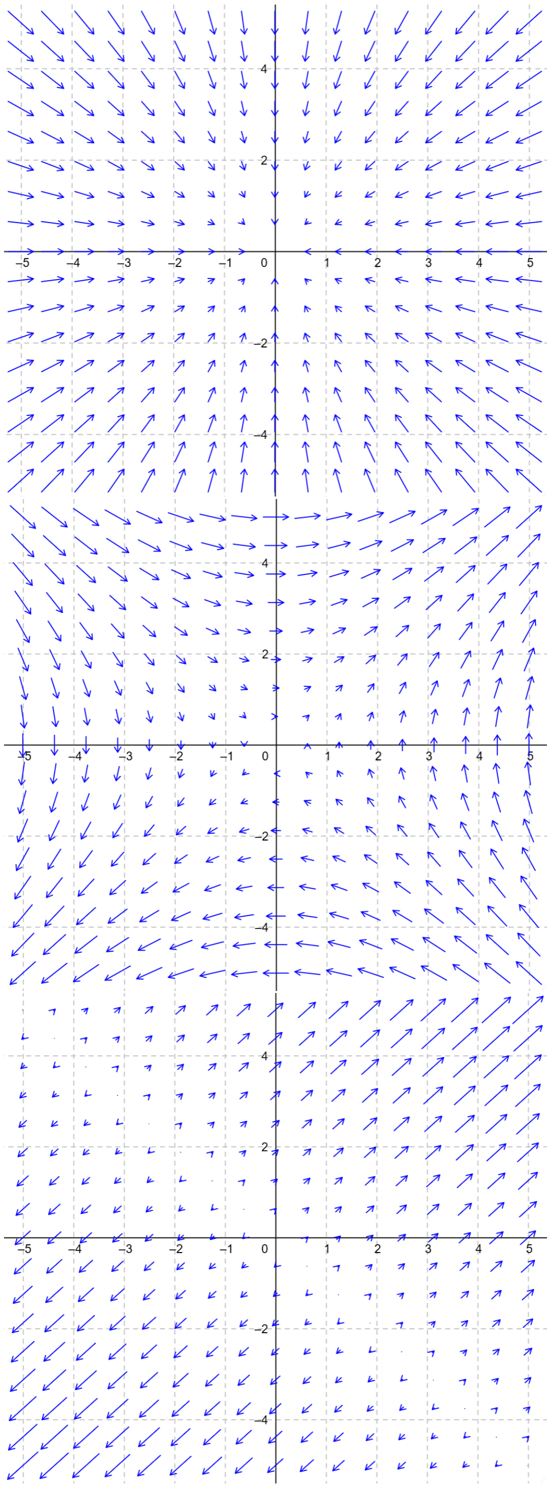

Figure 1: Plots of the vector fields from Question 1, listed from a-c top to bottom.

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Question 3 Which of the following 1-forms are exact? If they are exact, find a 0-form f such that $\omega = \mathbf{d}f$. If they are not exact, prove no such 0-form exists.

(1) $\omega = \sin y \mathbf{d}x + x \cos y \mathbf{d}y$

(2) $\omega = 2x\mathbf{d}x + 3y^2\mathbf{d}y$

(3) $\omega = x^3\mathbf{d}y$

Solution. (1) ω is exact: the 0-form $f = x \sin y$ satisfies $\omega = \mathbf{d}f$.

(2) ω is exact: the 0-form $x^2 + y^3$ satisfies $\omega = \mathbf{d}f$.

(3) ω is not exact: if there exists an f such that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = x^3$, then

$$3x^2 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0,$$

which is a contradiction.

□

Question 4 Consider the 0-form $f = x^3 - y^3$ and the vector field $\mathbf{W} = -y\mathbf{e}_x + x\mathbf{e}_y$.

(1) Calculate the differential of f.

(2) Compute $\mathbf{d}f(-, \mathbf{W})$.

(3) Using your previous answer, find all points in $\mathbb{A}^2$ where the directional derivative $D_{\mathbf{W}}f$ of f along $\mathbf{W}$ is zero.

Solution. (1) By the definition of the differential, we have $\mathbf{d}f = 3x^2\mathbf{d}x - 3y^2\mathbf{d}y$.

(2) Using the definition of applying 1-forms to vector fields from the notes, we get

$$\mathbf{d}f(-, \mathbf{W}) = (3x^2) \cdot -y + (-3y^2) \cdot x = -3xy(x + y).$$

(3) As $D_{\mathbf{W}}f = \mathbf{d}f(-, \mathbf{W}) = -3xy(x + y)$, we have

$$D_{\mathbf{W}}f = 0 \iff \begin{cases} x = 0 \\ y = 0 \\ x = -y \end{cases}.$$

□

2 Extra Exercises

Question 5 We shall show that 1-form computations do not depend on the coordinate system by considering a computation in polar coordinates (r, θ).

(1) Consider the vector field $\mathbf{W} = -y\mathbf{e}_x + x\mathbf{e}_y$. Show that we can rewrite $\mathbf{W}$ in polar coordinates as

$$\mathbf{W} = \mathbf{e}_\theta.$$

(2) Polar coordinates have the elementary 1-forms $\mathbf{d}r, \mathbf{d}\theta$ where for a vector $\mathbf{v} = [v_r, v_\theta]^\mathsf{T}$ we have

$$\mathbf{d}r(\mathbf{v}) = v_r, \quad \mathbf{d}\theta(\mathbf{v}) = v_\theta.$$

Furthermore, they are related to the elementary forms $\mathbf{d}x, \mathbf{d}y$ by

$$\mathbf{d}x = \frac{\partial x}{\partial r}\mathbf{d}r + \frac{\partial x}{\partial \theta}\mathbf{d}\theta, \quad \mathbf{d}y = \frac{\partial y}{\partial r}\mathbf{d}r + \frac{\partial y}{\partial \theta}\mathbf{d}\theta.$$

Show that we can rewrite the 1-form $\omega = x\mathbf{d}x + y\mathbf{d}y$ in polar coordinates as

$$\omega = r\mathbf{d}r.$$

(3) Calculate $\omega(-, \mathbf{W})$ in polar coordinates. How does your solution relate to the same calculation in Cartesian coordinates? (Example 2.29 in the notes)

Solution. (1) We can write $\mathbf{e}_x, \mathbf{e}_y$ in terms of $\mathbf{e}_r, \mathbf{e}_\theta$ in a similar way as (5b) on Exercise sheet 5:

$$\mathbf{e}_x = \frac{\partial r}{\partial x}\mathbf{e}_r + \frac{\partial \theta}{\partial x}\mathbf{e}_\theta = \frac{x}{\sqrt{x^2+y^2}}\mathbf{e}_r + \frac{-y}{x^2+y^2}\mathbf{e}_\theta = \cos\theta\mathbf{e}_r - \frac{\sin\theta}{r}\mathbf{e}_\theta \tag{1}$$

$$\mathbf{e}_y = \frac{\partial r}{\partial y}\mathbf{e}_r + \frac{\partial \theta}{\partial y}\mathbf{e}_\theta = \frac{y}{\sqrt{x^2+y^2}}\mathbf{e}_r + \frac{x}{x^2+y^2}\mathbf{e}_\theta = \sin\theta\mathbf{e}_r + \frac{\cos\theta}{r}\mathbf{e}_\theta \tag{2}$$

We can substitute these values to write our vector field $\mathbf{W}$ in polar coordinates:

$$\begin{aligned}\mathbf{W} &= -r\sin\theta\left(\cos\theta\mathbf{e}_r - \frac{\sin\theta}{r}\mathbf{e}_\theta\right) + r\cos\theta\left(\sin\theta\mathbf{e}_r + \frac{\cos\theta}{r}\mathbf{e}_\theta\right)\\ &= (-r\sin\theta\cos\theta + r\sin\theta\cos\theta)\mathbf{e}_r + (\sin^2\theta + \cos^2\theta)\mathbf{e}_\theta\\ &= \mathbf{e}_\theta\end{aligned}$$

(2)

$$\begin{aligned}\mathbf{d}x &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\\ &= \cos\theta dr + (-r\sin\theta)d\theta \end{aligned}\tag{3}$$

$$\begin{aligned}\mathbf{d}y &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\\ &= \sin\theta dr + r\cos\theta d\theta \end{aligned}\tag{4}$$

A similar substitution to part (a) gives

$$\begin{aligned}\omega &= x\mathbf{d}x + y\mathbf{d}y\\ &= r\cos\theta(\cos\theta dr + (-r\sin\theta)d\theta) + r\sin\theta(\sin\theta dr + r\cos\theta d\theta)\\ &= (r\cos^2\theta + r\sin^2\theta)dr + (-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta)d\theta\\ &= rdr\end{aligned}$$

(3) Given a 1-form $\omega = g_r\mathbf{d}r + g_\theta\mathbf{d}\theta$ and a vector field $\mathbf{W} = h_r\mathbf{e}_r + h_\theta\mathbf{e}_\theta$, we calculate $\omega(-, \mathbf{W})$ that same way as in Cartesian coordinates:

$$\omega(-, \mathbf{W}) = g_r \cdot h_r + g_\theta \cdot h_\theta = r \cdot 0 + 0 \cdot 1 = 0$$

This is the same answer as we got in the notes for Cartesian coordinates, as these calculations are invariant under change of coordinates. The conclusion is far more obvious now: $\omega(-, \mathbf{W})$ is how a vector field dependent on $\mathbf{e}_\theta$ changes with respect to a 1-form in terms of r, and clearly these two have no interaction as they are different variables.

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