Exercises 5

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

For all questions, we work in Cartesian coordinates $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in \mathbb{A}^2$, where each tangent space has basis $(\mathbf{e}_x, \mathbf{e}_y) \subset \mathbb{E}^2$.

Question 1 Calculate the area of the parallelogram $▱(\mathbf{a}, \mathbf{b})$, formed by the following vectors, where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis.

(1) $\mathbf{a} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$;

(2) $\mathbf{a} = 5\mathbf{e}_1 + 8\mathbf{e}_2 + 4\mathbf{e}_3$ and $\mathbf{b} = 10\mathbf{e}_1 + 16\mathbf{e}_2 + 8\mathbf{e}_3$.

Solution. For calculation of length the orientation is unimportant: we can use the vector product with either orientation.

(1) $$\text{Area}(▱(\mathbf{a}, \mathbf{b})) = \|\mathbf{a} \times \mathbf{b}\| = \left(\left(\det\begin{bmatrix}2 & 3\\1 & 1\end{bmatrix}\right)^2 + \left(\det\begin{bmatrix}2 & 3\\1 & 1\end{bmatrix}\right)^2 + \left(\det\begin{bmatrix}2 & 2\\1 & 1\end{bmatrix}\right)^2\right)^{1/2} = \sqrt{2}.$$

(2) $\mathbf{b} = 2\mathbf{a}$: the vectors are collinear, the vector product is zero and the area is zero.

□

Question 2 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$ and let P be a linear operator with the matrix: $[P]_{\mathcal{B}} = \left[\begin{smallmatrix}5 & -1\\2 & 2\end{smallmatrix}\right]$. (See Exercise Sheet 2 Question 8.)

(1) Calculate the area of the parallelogram $▱(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2)$.

(2) Recall the $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$ is an eigenvector of P with eigenvalue 4; and $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2$ is an eigenvector of P with eigenvalue 3. Calculate the area of the image parallelogram $▱(P(\mathbf{a}), P(\mathbf{b}))$ without using the determinant.

(3) Compare your answers to parts (1) and (2) with the $\det P$.

Solution. (1) We can calculate the area using the 2×2 determinant:

$$\text{Area} = \left|\det\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\right| = |2 - 1| = 1.$$

(2) As $\mathbf{a}$ and $\mathbf{b}$ are eigenvalues we know $P(\mathbf{a}) = 4\mathbf{e}_1 + 4\mathbf{e}_2$; and $P(\mathbf{b}) = 3\mathbf{e}_1 + 6\mathbf{e}_2$.

$$\text{Area} = \left|\det\begin{bmatrix}4 & 4\\3 & 6\end{bmatrix}\right| = |24 - 12| = 12.$$

(3) $\det P = 5 \times 2 - (-1) \times 2 = 12$. The determinant of the operator is 12 and the image parallelogram is $\det P = 12$ times bigger than the original parallelogram.

□

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis for $\mathbb{E}^3$. Let P be the linear operator with matrix $[P]_{\mathcal{B}} = \begin{bmatrix}1 & 3 & -4\\2 & 4 & -4\\0 & 0 & 1\end{bmatrix}$.

(1) Calculate the volume of the parallelepiped formed by the vectors $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_3$; $\mathbf{b} = 4\mathbf{e}_2 + 3\mathbf{e}_3$; and $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2$.

(2) Calculate $P(\mathbf{a})$, $P(\mathbf{b})$ and $P(\mathbf{c})$.

(3) What is the volume of the parallelepiped $▱(P(\mathbf{a}), P(\mathbf{b}), P(\mathbf{c}))$? (without using the determinant)

(4) Compare the volumes from parts (1) and (3) with the determinant of P.

Solution. (1) The volume of the parallelepiped is given by the (absolute value of) 3×3 determinant

$$\text{Vol} = \left|\det\begin{bmatrix}2 & 0 & 1\\0 & 4 & 3\\1 & -1 & 0\end{bmatrix}\right| = |2 \times 3 + 1 \times (-4)| = 2.$$

(2)
$$P(\mathbf{a}) = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad P(\mathbf{b}) = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \qquad P(\mathbf{c}) = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

(3)
$$\mathrm{Vol}\Big(▱(P(\mathbf{a}), P(\mathbf{b}), P(\mathbf{c}))\Big) = \left| \det \begin{bmatrix} -2 & 0 & 1 \\ 0 & 4 & 3 \\ -2 & -2 & 0 \end{bmatrix} \right| = |-2 \times 6 + 1 \times 8| = 4.$$

(4) Calculating the determinant using the last row:
$$\det P = \det \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = -2.$$

The volume of the parallelepiped is scaled by $2 = |\det P|$.

□

2 Extra Exercises

Question 4 Let $\mathcal{B}$, P, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be as in Question 3.

(1) What is the area of the parallelograms:

(a) $▱(\mathbf{a}, \mathbf{b})$; (b) $▱(\mathbf{a}, \mathbf{c})$?

(2) Compare these to the areas of the image parallelograms:

(a) $▱(P(\mathbf{a}), P(\mathbf{b}))$; (b) $▱(P(\mathbf{a}), P(\mathbf{c}))$?

(3) Can you say anything about the determinant of a linear operator acting on a 3-dimensional space and the scaling of parallelograms?

(4) Can you find a linear operator that fixes the volume of parallelepipeds but scales the area of some parallelogram by λ?

Solution. (1)

(a)
$$\begin{aligned}\text{Area}\big(▱(\mathbf{a}, \mathbf{b})\big) &= \|\mathbf{a} \times \mathbf{b}\| \\ &= \left\| \mathbf{e}_1 \det \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} - \mathbf{e}_2 \det \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \mathbf{e}_3 \det \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \right\| \\ &= \big(16 + 36 + 64\big)^{1/2} \\ &= 2\sqrt{29}\end{aligned}$$

(b)
$$\begin{aligned}\text{Area}\big(▱(\mathbf{a}, \mathbf{c})\big) &= \left\| \mathbf{e}_1 \det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \mathbf{e}_2 \det \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + \mathbf{e}_3 \det \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \right\| \\ &= \big(1 + 1 + 4\big)^{1/2} \\ &= \sqrt{6}\end{aligned}$$

(2)

(a)
$$\begin{aligned}\text{Area}\big(▱(P(\mathbf{a}), P(\mathbf{b}))\big) &= \left\| \mathbf{e}_1 \det \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} - \mathbf{e}_2 \det \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} + \mathbf{e}_3 \det \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \right\| \\ &= \big(16 + 36 + 64\big)^{1/2} \\ &= 2\sqrt{29}\end{aligned}$$

(b)
$$\begin{aligned}\text{Area}\big(▱(P(\mathbf{a}), P(\mathbf{c}))\big) &= \left\| \mathbf{e}_1 \det \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} - \mathbf{e}_2 \det \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} + \mathbf{e}_3 \det \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} \right\| \\ &= \big(4 + 4 + 16\big)^{1/2} \\ &= \sqrt{24} \\ &= 2\sqrt{6}\end{aligned}$$

(3) The linear operator fixes the area of the first parallelogram, but doubles the area of the second. In general the determinant of linear operator acting on a 3-dimensional Euclidean space controls the scaling of volumes, but says nothing about the scaling of areas (or lines).

(4) There are many choices: consider a linear operator with matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}.$$

This has determinant 1 and so the volumes of parallelepipeds are fixed by the operator. The parallelogram (actually a rectangle) formed by the vectors $[1\ 0\ 0]^\mathsf{T}$ and $[0\ 1\ 0]^\mathsf{T}$ has area 1, but is mapped to the parallelogram formed by $[\lambda\ 0\ 0]^\mathsf{T}$ and $[0\ 1\ 0]^\mathsf{T}$ with area λ.

□

Question 5 Consider the set of points

$$L = \left\{ \begin{pmatrix} x \\ mx + c \end{pmatrix} \in \mathbb{A}^2 \ \mid \ x \in \mathbb{R} \right\}, \ m, c \in \mathbb{R}.$$

Show that L is an affine space.

[Hint: As L forms a line, the associated vector space should be one-dimensional.]

Solution. Note that L is a line in the plane with gradient m. We would like to find a vector space V such that we have an addition map

$$\begin{aligned} L \times V &\to L \\ (\mathbf{P}, \mathbf{v}) &\mapsto \mathbf{P} + \mathbf{v}\,, \end{aligned}$$

in particular $\mathbf{P} + \mathbf{v}$ is always a point on our line L. We consider the vector space

$$V = \text{span}\left(\begin{bmatrix} 1 \\ m \end{bmatrix} \right),$$

whose vectors span the line with gradient m going through the origin, *i.e.*, it is parallel to L. Then for any $\mathbf{P} \in L, \mathbf{v} \in V$, their sum is contained in L:

$$\mathbf{P} = \begin{pmatrix} x \\ mx + c \end{pmatrix}, \ \mathbf{v} = \begin{bmatrix} \alpha \\ m\alpha \end{bmatrix} \Rightarrow \mathbf{P} + \mathbf{v} = \begin{pmatrix} x + \alpha \\ m(x + \alpha) + c \end{pmatrix} \in L$$

Therefore this addition map is well defined.

It remains to check the three axioms for affine space:

Associativity:

$$\begin{aligned} \mathbf{P} + (\mathbf{v} + \mathbf{w}) &= \begin{pmatrix} x \\ mx + c \end{pmatrix} + \begin{bmatrix} \alpha + \beta \\ m(\alpha + \beta) \end{bmatrix} = \begin{pmatrix} x + \alpha + \beta \\ m(x + \alpha + \beta) + c \end{pmatrix} \\ (\mathbf{P} + \mathbf{v}) + \mathbf{w} &= \begin{pmatrix} x + \alpha \\ m(x + \alpha) + c \end{pmatrix} + \begin{bmatrix} \beta \\ m\beta \end{bmatrix} = \begin{pmatrix} x + \alpha + \beta \\ m(x + \alpha + \beta) + c \end{pmatrix} \end{aligned}$$

Zero:

$$\mathbf{P} + \mathbf{0} = \begin{pmatrix} x \\ mx + c \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix}$$

Vectors between points: Let P, Q be points in L, we can consider the vector

$$\begin{aligned} \mathbf{v} = \mathbf{Q} - \mathbf{P} &= \begin{pmatrix} y \\ my + c \end{pmatrix} - \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{bmatrix} y - x \\ m(y - x) \end{bmatrix} \in V \\ \Rightarrow \mathbf{P} + \mathbf{v} &= \begin{pmatrix} x \\ mx + c \end{pmatrix} + \begin{bmatrix} y - x \\ m(y - x) \end{bmatrix} = \begin{pmatrix} y \\ my + c \end{pmatrix} = \mathbf{Q} \end{aligned}$$

□

Question 6 Fix Cartesian coordinates (x, y) on $\mathbb{A}^2$ with respect to an origin $\mathbf{O}$ and orthonormal basis $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$.

(1) One way to define polar coordinates (r, θ) on $\mathbb{A}^2$ is via the map

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{A}^2 \\ (r, \theta) &\mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \end{aligned}$$

Show this map is a coordinate system. *[Hint: It may be helpful to recall the following trigonometric identities:]*

$$\sin(\arctan(z)) = \frac{z}{\sqrt{1 + z^2}}, \quad \cos(\arctan(z)) = \frac{1}{\sqrt{1 + z^2}}$$

(2) Polar coordinates give rise to a different basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ for the tangent spaces $T_{\mathbf{P}}\left(\mathbb{A}^2\right)$ via the equations:

$$\mathbf{e}_r = \frac{\partial x}{\partial r}\mathbf{e}_x + \frac{\partial y}{\partial r}\mathbf{e}_y, \quad \mathbf{e}_\theta = \frac{\partial x}{\partial \theta}\mathbf{e}_x + \frac{\partial y}{\partial \theta}\mathbf{e}_y.$$

Write down the basis vectors in terms of $\mathbf{e}_x, \mathbf{e}_y$. Is this basis orthonormal?

(3) Describe how the basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ for $T_{\mathbf{P}}\left(\mathbb{A}^2\right)$ varies as we vary $\mathbf{P}$. What is the connection between (r, θ) and $(\mathbf{e}_r, \mathbf{e}_\theta)$?

Solution. (1) The map defining polar coordinates is a coordinate system if it is surjective. Therefore for some point $(x, y) \in \mathbb{A}^2$, we must find reals r, θ such that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We first note that if the above equations hold, we have

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2,$$

therefore we can pick $r = \sqrt{x^2+y^2}$. Furthermore, we can find a possible expression for θ via

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta,$$

therefore we can set $\theta = \arctan(\frac{y}{x})$.

Finally, let us check these values do give us x and y:

$$r\cos\theta = \sqrt{x^2+y^2}\cos\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{\sqrt{x^2+y^2}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = x$$

$$r\sin\theta = \sqrt{x^2+y^2}\sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{\left(\frac{y}{x}\right)\sqrt{x^2+y^2}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = y$$

(2) As $x = r\cos\theta, y = r\sin\theta$ we have

$$\mathbf{e}_r = \frac{\partial x}{\partial r}\mathbf{e}_x + \frac{\partial y}{\partial r}\mathbf{e}_y = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y$$

$$\mathbf{e}_\theta = \frac{\partial x}{\partial \theta}\mathbf{e}_x + \frac{\partial y}{\partial \theta}\mathbf{e}_y = -r\sin\theta\mathbf{e}_x + r\cos\theta\mathbf{e}_y$$

To check if this basis is orthonormal, we calculate the following inner products

$$\begin{aligned}
\langle \mathbf{e}_r, \mathbf{e}_\theta\rangle &= \langle \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, -r\sin\theta\mathbf{e}_x + r\cos\theta\mathbf{e}_y\rangle \\
&= -r\sin\theta\cos\theta\langle \mathbf{e}_x, \mathbf{e}_x\rangle + r\cos^2\theta\langle \mathbf{e}_x, \mathbf{e}_y\rangle + -r\sin^2\theta\langle \mathbf{e}_y, \mathbf{e}_x\rangle + r\sin\theta\cos\theta\langle \mathbf{e}_y, \mathbf{e}_y\rangle \\
&= -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0 \\
\langle \mathbf{e}_r, \mathbf{e}_r\rangle &= \langle \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y\rangle \\
&= \cos^2\theta\langle \mathbf{e}_x, \mathbf{e}_x\rangle + \sin\theta\cos\theta\langle \mathbf{e}_x, \mathbf{e}_y\rangle + \sin\theta\cos\theta\langle \mathbf{e}_y, \mathbf{e}_x\rangle + \sin^2\theta\langle \mathbf{e}_y, \mathbf{e}_y\rangle \\
&= \cos^2\theta + \sin^2\theta = 1 \\
\langle \mathbf{e}_\theta, \mathbf{e}_\theta\rangle &= \langle -r\sin\theta\mathbf{e}_x + r\cos\theta\mathbf{e}_y, -r\sin\theta\mathbf{e}_x + r\cos\theta\mathbf{e}_y\rangle \\
&= r^2\sin^2\theta\langle \mathbf{e}_x, \mathbf{e}_x\rangle - r^2\sin\theta\cos\theta\langle \mathbf{e}_x, \mathbf{e}_y\rangle - r^2\sin\theta\cos\theta\langle \mathbf{e}_y, \mathbf{e}_x\rangle + r^2\cos^2\theta\langle \mathbf{e}_y, \mathbf{e}_y\rangle \\
&= r^2(\cos^2\theta + \sin^2\theta) = r^2
\end{aligned}$$

The two basis vectors are orthogonal, and $\mathbf{e}_r$ is unit length but $\mathbf{e}_\theta$ is not unit length. Therefore $(\mathbf{e}_r, \mathbf{e}_\theta)$ is not an orthonormal basis.

(3) The basis vector $\mathbf{e}_r$ points in the radial direction, *i.e.*, away from the origin $\mathbf{O}$. As we vary $\mathbf{P}$, its direction changes by how much the angle θ changes. Its magnitude never changes as it is always unit length.

The basis vector $\mathbf{e}_\theta$ points in the orthogonal direction to $\mathbf{e}_\mathbf{r}$, *i.e.*, the direction in which rotations occur. As we vary $\mathbf{P}$, its direction also changes by how much the angle θ changes. Its magnitude increases as r increases. This is because a small change in θ moves $\mathbf{P}$ less distance when r is small than when r is large.

□