Exercises 4

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Define a linear operator P on $\mathbb{E}^3$ by

$$P(\mathbf{e}) = \mathbf{e}$$
$$P(\mathbf{f}) = \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}$$
$$P(\mathbf{g}) = -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}$$

(1) Write down the matrix $[P]_{\mathcal{B}}$.

(2) Show P is an orthogonal operator that preserves orientation.

(3) Find the axis and angle of rotation.

Solution. (1) The matrix for P in terms of $\mathcal{B}$ is

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

(2) A standard calculation shows $[P]_{\mathcal{B}}$ is an orthogonal matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore P is orthogonal. Furthermore, $\det(P) = \det([P]_{\mathcal{B}}) = 1$ therefore it preserves orientation.

(3) By Euler's Theorem, we know P is a rotation operator. We could calculate the eigenvectors from $[P]_{\mathcal{B}}$ to find the axis of rotation, however by definition of P

$$P(\mathbf{e}) = \mathbf{e}.$$

Therefore the axis of rotation is $L_{\mathbf{e}} = \operatorname{span}(\mathbf{e})$.

The trace of P is the sum of elements on the diagonal of $[P]_{\mathcal{B}}$:

$$\operatorname{Tr}(P) = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 1 + \sqrt{2} \Rightarrow \varphi = \arccos\left(\frac{\sqrt{2}}{2}\right) = \pm\frac{\pi}{4}.$$

However, as our matrix $[P]_{\mathcal{B}}$ is in a nice form, we can compute the sign of this angle:

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$
$$\Rightarrow \cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}$$
$$\Rightarrow \varphi = \frac{\pi}{4}$$

□

Question 2 Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Consider the linear operator P_1 in $\mathbb{E}^3$ defined by

$$P_1(\mathbf{e}) = \mathbf{f} \qquad P_1(\mathbf{f}) = \mathbf{e} \qquad P_1(\mathbf{g}) = \mathbf{g}.$$

Also consider the linear operator P_2 that is the reflection operator in the plane spanned by $\mathbf{e}$ and $\mathbf{f}$.

(1) State whether P_1 and P_2 preserves orientation.

(2) Define the operator $P = P_1 \circ P_2$ that applies the operator P_2 followed by the operator P_1. Does P preserve orientation?

(3) Show that P is a rotation operator. Find its axis and angle of rotation.

Solution. (1) The matrix for P_1 in terms of $\mathcal{B}$ is

$$[P_1]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

whose determinant is -1, therefore P_1 does not preserve orientation.

Note that by orthonormality, $\mathbf{g}$ is a normal vector to the plane spanned by $\mathbf{e}, \mathbf{f}$. Therefore P_2 fixes $\mathbf{e}$ and $\mathbf{f}$ but sends $\mathbf{g} \mapsto -\mathbf{g}$:

$$[P_2]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The determinant of this matrix is -1 so P_2 also doesn't preserves orientation.

(2) The matrix for P is the product of the matrices for P_1 and P_2. We can quickly see whether P preserves orientation by computing

$$\det([P]_{\mathcal{B}}) = \det([P_1]_{\mathcal{B}}) \det([P_2]_{\mathcal{B}}) = 1$$

and so P does preserve orientation. However, we need the matrix for P in the next part anyway, so we could directly calculate the determinant of P from this.

$$[P]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(3) P preserves orientation, therefore suffices to show it is orthogonal, i.e.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$[P]_{\mathcal{B}}$ is an orthogonal matrix, therefore P is a rotation operator by Euler's Theorem.

To calculate its eigenvectors with eigenvalue 1, we compute

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} -x + y &= 0 \\ x - y &= 0. \\ -2z &= 0 \end{aligned}$$

Therefore $\mathbf{e} + \mathbf{f}$ is an eigenvector and so the axis of rotation is $L_{\mathbf{e}+\mathbf{f}} = \mathrm{span}(\mathbf{e} + \mathbf{f})$.

The trace of P is

$$\mathrm{Tr}(P) = -1 = 1 + 2\cos\varphi \quad \Rightarrow \quad \varphi = \arccos(-1) = \pi.$$

Therefore P is a rotation about $L_{\mathbf{e}+\mathbf{f}}$ by the angle π. □

2 Extra Exercises

Question 3 Let $\mathbf{n} \in \mathbb{E}^3$ be a unit vector and consider the following operators on $\mathbb{E}^3$

$$P_1(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n}, \quad P_2(x) = 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n} - \mathbf{x}.$$

(1) Show both operators are orthogonal.

(2) Show the first operator is a reflection, along with the plane it is a reflection through.

(3) Show the second operator is a rotation. Find its axis and angle of rotation.

[(Almost) all of this question can be done without writing a matrix! You may find it helpful to consider the plane $H_{\mathbf{n}}$ whose normal vector is $\mathbf{n}$:[1]*]*

$$H_{\mathbf{n}} = \left\{\mathbf{x} \in \mathbb{E}^3 \;\middle|\; \langle \mathbf{n}, \mathbf{x}\rangle = 0\right\}$$

Solution. (1) To show P_1 is orthogonal, we consider the inner product

$$\begin{aligned}\langle P_1(\mathbf{x}), P_1(\mathbf{y})\rangle &= \langle \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n}, \mathbf{y} - 2\langle \mathbf{n}, \mathbf{y}\rangle \mathbf{n}\rangle \\ &= \langle \mathbf{x}, \mathbf{y}\rangle - 2\langle \mathbf{n}, \mathbf{x}\rangle\langle \mathbf{n}, \mathbf{y}\rangle - 2\langle \mathbf{n}, \mathbf{y}\rangle\langle \mathbf{n}, \mathbf{x}\rangle + 4\langle \mathbf{n}, \mathbf{x}\rangle\langle \mathbf{n}, \mathbf{y}\rangle\langle \mathbf{n}, \mathbf{n}\rangle.\end{aligned}$$

As $\mathbf{n}$ is a unit vector, $\langle \mathbf{n}, \mathbf{n}\rangle = 1$ and so this simplifies to $\langle P_1(\mathbf{x}), P_1(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$. Therefore P_1 is orthogonal.

As $P_2 = -P_1$, a similar calculation shows that P_2 is also orthogonal.

(2) Applying P_1 to $\mathbf{n}$, we see

$$P_1(\mathbf{n}) = \mathbf{n} - 2\langle \mathbf{n}, \mathbf{n}\rangle \mathbf{n} = -\mathbf{n}.$$

Furthermore, applying P_1 to an element $\mathbf{x} \in H_{\mathbf{n}}$ we see

$$P_1(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n} = \mathbf{x} - 0 \cdot \mathbf{n} = \mathbf{x}.$$

Therefore P_1 leaves $H_{\mathbf{n}}$ fixed but maps its normal vector $\mathbf{n}$ to $-\mathbf{n}$. Therefore P_1 is a reflection in the plane $H_{\mathbf{n}}$.

(3) As P_1 is a reflection, it reverses orientation therefore $\det(P_1) < 0$. Explicitly, if $\mathbf{f}, \mathbf{g}$ are an orthonormal basis for the plane $H_{\mathbf{n}}$, then P_1 maps the orthonormal basis $(\mathbf{n}, \mathbf{f}, \mathbf{g})$ to $(-\mathbf{n}, \mathbf{f}, \mathbf{g})$, reversing the orientation. As $P_2 = -P_1$, we have $\det(P_2) = -\det(P_1) > 0$ and so P_2 preserves orientation. Furthermore it is orthogonal and so must be a rotation by Euler's Theorem.

We note that

$$P_2(\mathbf{n}) = 2\langle \mathbf{n}, \mathbf{n}\rangle \mathbf{n} - \mathbf{n} = \mathbf{n}$$

and so the axis of rotation is $L_{\mathbf{n}}$. Considering a vector $\mathbf{y} \in H_{\mathbf{n}}$, we see that

$$P_2(\mathbf{y}) = 2\langle \mathbf{n}, \mathbf{y}\rangle \mathbf{n} - \mathbf{y} = -\mathbf{y},$$

therefore P_2 rotates the plane $H_{\mathbf{n}}$ by the angle π.

□

Question 4 † In this question we will prove the determinant formula satisfies the axioms of a vector product. Fix an orthogonal basis $\mathcal{B}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{E}^3$, which we use to define the orientation of $\mathbb{E}^3$. For any two vectors $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ define the function $- \times -\colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ by the determinant formula:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

(VP-AC) Show, or explain why, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

((VP-⊥)) (1) Show that $\langle \mathbf{v} \times \mathbf{w}, \mathbf{v}\rangle = 0$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

[1] In general, any plane in $\mathbb{E}^3$ (or $(n-1)$-dimensional *hyperplane* in $\mathbb{E}^n$) can be written as the set of vectors orthogonal to a normal vector $\mathbf{n}$.

(2) Using (VP-AC) deduce that $\langle \mathbf{v} \times \mathbf{w},\, \mathbf{w} \rangle = 0$.

(VP-Lin) (1) Show the following identity holds:

$$\det \begin{bmatrix} \lambda a + \mu a' & \lambda b + \mu b' \\ c & d \end{bmatrix} = \lambda \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mu \det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}.$$

(2) Prove that $(\lambda \mathbf{v} + \mu \mathbf{w}) \times \mathbf{x} = \lambda(\mathbf{v} \times \mathbf{x}) + \mu(\mathbf{w} \times \mathbf{x})$ for all vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

(VP-Len) Recall the identity of Lemma 1.88: $\|\mathbf{v} \times \mathbf{w}\|^2 + \langle \mathbf{v},\, \mathbf{w} \rangle^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$.

Using this formula, or otherwise, prove $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|$ for all perpendicular vectors $\mathbf{v}$ and $\mathbf{w}$.

(VP-O) Let $v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$, be linearly independent vectors.

(1) Let $\mathcal{C} = (\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w})$, which you may assume is a basis for $\mathbb{E}^3$. Write down the transition matrix T, from basis $\mathcal{B}$ to basis $\mathcal{C}$.

(2) Calculate a formula for the determinant of T and deduce that $\mathcal{B}$ and $\mathcal{C}$ have the same orientation.
It may be helpful to use the 3^{rd} *column to calculate the determinant:*

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} - f \det \begin{bmatrix} a & b \\ g & h \end{bmatrix} + i \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

Finally: Deduce that $- \times -$ is a vector product as defined in Definition 1.82.

Solution. Throughout, we shall let $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ and $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.

(VP-AC) Anticommutative This follows immediately from the fact that swapping a pair of rows in a matrix, switches the sign of the determinant. One can also write down the two determinants and compare.

(VP-⊥) Perpendicular

(1)

$$\begin{aligned} \langle \mathbf{v} \times \mathbf{w},\, \mathbf{v} \rangle &= \langle (v_2w_3 - v_3w_2)\mathbf{e}_1,\, \mathbf{v} \rangle \\ &\quad + \langle (v_3w_1 - v_1w_3)\mathbf{e}_2,\, \mathbf{v} \rangle \\ &\quad + \langle (v_1w_2 - v_2w_1)\mathbf{e}_3,\, \mathbf{v} \rangle && \text{(linearity of } \langle -, - \rangle) \\ &= \langle (v_2w_3 - v_3w_2)\mathbf{e}_1,\, v_1\mathbf{e}_1 \rangle \\ &\quad + \langle (v_3w_1 - v_1w_3)\mathbf{e}_2,\, v_2\mathbf{e}_2 \rangle \\ &\quad + \langle (v_1w_2 - v_2w_1)\mathbf{e}_3,\, v_3\mathbf{e}_3 \rangle && \text{(the basis is orthonormal)} \\ &= \cancel{v_2w_3v_1} - \cancel{v_3w_2v_1} + \cancel{v_3w_1v_2} \\ &\quad - \cancel{v_1w_3v_2} + \cancel{v_1w_2v_3} - \cancel{v_2w_1v_3} = 0 \end{aligned}$$

(2)

$$\begin{aligned} \langle \mathbf{v} \times \mathbf{w},\, \mathbf{w} \rangle &= -\langle \mathbf{w} \times \mathbf{v},\, \mathbf{w} \rangle && \text{Using (VP-AC)} \\ &= 0 && \text{Using part (1)} \end{aligned}$$

(VP-Lin) Linearity

(1)

$$\begin{aligned} \det \begin{bmatrix} \lambda a + \mu a' & \lambda b + \mu b' \\ c & d \end{bmatrix} &= (\lambda a + \mu a')d - (\lambda b + \mu b')c \\ &= \lambda(ad - bc) + \mu(a'd - b'c) \\ &= \lambda \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mu \det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \end{aligned}$$

(2)
$$\begin{aligned}
(\lambda\mathbf{v}+\mu\mathbf{w})\times\mathbf{x} &= \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \lambda v_1+\mu w_1 & \lambda v_2+\mu w_2 & \lambda v_3+\mu w_3 \\ x_1 & x_2 & x_3 \end{bmatrix} \\
&= \mathbf{e}_1 \det\begin{bmatrix} \lambda v_2+\mu w_2 & \lambda v_3+\mu w_3 \\ x_2 & x_3 \end{bmatrix} - \mathbf{e}_2 \det\begin{bmatrix} \lambda v_1+\mu w_1 & \lambda v_3+\mu w_3 \\ x_1 & x_3 \end{bmatrix} + \\
&\quad \mathbf{e}_3 \det\begin{bmatrix} \lambda v_1+\mu w_1 & \lambda v_2+\mu w_2 \\ x_1 & x_2 \end{bmatrix} \\
&= \lambda \left(\mathbf{e}_1 \det\begin{bmatrix} v_2 & v_3 \\ x_2 & x_3 \end{bmatrix} - \mathbf{e}_2 \det\begin{bmatrix} v_1 & v_3 \\ x_1 & x_3 \end{bmatrix} \mathbf{e}_3 \det\begin{bmatrix} v_1 & v_2 \\ x_1 & x_2 \end{bmatrix}\right) + \\
&\quad \mu \left(\mathbf{e}_1 \det\begin{bmatrix} w_2 & w_3 \\ x_2 & x_3 \end{bmatrix} - \mathbf{e}_2 \det\begin{bmatrix} w_1 & w_3 \\ x_1 & x_3 \end{bmatrix} \mathbf{e}_3 \det\begin{bmatrix} w_1 & w_2 \\ x_1 & x_2 \end{bmatrix}\right) \\
&= \lambda \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ x_1 & x_2 & x_3 \end{bmatrix} + \mu \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ w_1 & w_2 & w_3 \\ x_1 & x_2 & x_3 \end{bmatrix} \\
&= \lambda(\mathbf{v}\times\mathbf{x}) + \mu(\mathbf{w}\times\mathbf{x})
\end{aligned}$$

(VP-Len) Length For orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$ the inner product $\langle \mathbf{v}, \mathbf{w}\rangle = 0$, thus

$$\|\mathbf{v}\times\mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \quad \Rightarrow \quad \|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|.$$

(VP-O) Orientation

(1)
$$T = \begin{bmatrix} v_1 & w_1 & v_2w_3 - v_3w_2 \\ v_2 & w_2 & v_3w_1 - v_1w_3 \\ v_3 & w_3 & v_1w_2 - v_2w_1 \end{bmatrix}.$$

(2) We calculate the determinant using the right-hand column instead of the top row:

$$\begin{aligned}
\det T &= (v_2w_3 - v_3w_2)\det\begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} - (v_3w_1 - v_1w_3)\det\begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix} + \\
&\quad (v_1w_2 - v_2w_1)\det\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \\
&= (v_2w_3 - v_3w_2)^2 + (v_1w_3 - v_3w_1)^2 + (v_1w_2 - v_2w_1)^2 > 0
\end{aligned}$$

The determinant is greater than zero so the bases have same orientation. □

Finally: We have shown that function $- \times -$ satisfies all the axioms of Definition 1.82.

Question 5 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis in $\mathbb{E}^3$. Find a unit vector $\mathbf{n}$, such that the following conditions hold:

(1) $\mathbf{n}$ is orthogonal to $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$;

(2) $\mathbf{n}$ is orthogonal to $\mathbf{b} = \mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3$;

(3) The basis $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ has an orientation opposite to $\mathcal{B}$.

Express $\mathbf{n}$ using the basis $\mathcal{B}$.

Solution. Using the orientation of $\mathcal{B}$ for the vector product we let $\mathbf{v} = \mathbf{a}\times\mathbf{b}$. This is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and so $\mathbf{n}$ is proportional to $\mathbf{v}$. The vector $\mathbf{n}$ has length 1, so $\mathbf{n} = \frac{\pm\mathbf{v}}{\|\mathbf{v}\|}$. Finally, the basis $(\mathbf{a}, \mathbf{b}, \mathbf{v})$ has the same orientation as $\mathcal{B}$, so $(\mathbf{a}, \mathbf{b}, \frac{-\mathbf{v}}{\|\mathbf{v}\|})$ has the opposite orientation: the transition matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\|\mathbf{v}\|^{-1} \end{bmatrix}$. Therefore $\mathbf{n} = \frac{-\mathbf{v}}{\|\mathbf{v}\|}$.

$$\begin{aligned}
\mathbf{v} = \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} &= (4-9)\mathbf{e}_1 + (3-2)\mathbf{e}_2 + (3-2)\mathbf{e}_3 \\
&= -5\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\
\|\mathbf{v}\| &= \sqrt{25+1+1} = 3\sqrt{3} \\
\mathbf{n} = \frac{-\mathbf{v}}{\|\mathbf{v}\|} &= \frac{5\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3}{3\sqrt{3}}
\end{aligned}$$

□

Question 6 Elisa and Faraz calculate the vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{E}^3$. Elisa uses the orthonormal basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and Faraz use the orthonormal basis $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. The vectors expressed in these bases are:

$$[\mathbf{a}]_{\mathcal{E}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{E}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad [\mathbf{a}]_{\mathcal{F}} = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{F}} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

They both use the *determinant formula* for calculations:

$$\mathbf{a} \times \mathbf{b} \overset{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}}_{\text{Elisa's calculations}} \overset{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{bmatrix}}_{\text{Faraz's calculations}}.$$

(1) In what situations do their calculations give the same vector, and when do they give different answers?

(2) In each case what can you say about the linear operator P, that maps each vector $\mathbf{e}_i \mapsto \mathbf{f}_i$?

Solution. (1) If the bases $\mathcal{E}$ and $\mathcal{F}$ have the same orientation the answers will be the same. If they have an opposite orientation to each other then Faraz will calculate the negative of the vector calculated by Elisa.

(2) The operator P maps an orthonormal basis to an orthonormal basis, and hence is orthogonal ($\det P = \pm 1$). If Elisa and Faraz have the same answer, P preserves orientation and $\det P = 1$, otherwise P reverses the orientation and $\det P = -1$.

□