Exercises 3

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 (1) Group the following bases into equivalent classes with respect to orientation. That is, all bases in the same class must share an orientation and any pair in different classes must have a different orientation.

$$\mathcal{B}_1 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \qquad \mathcal{B}_2 = (\mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_x) \qquad \mathcal{B}_3 = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)$$
$$\mathcal{B}_4 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y - \mathbf{e}_x, \mathbf{e}_z) \qquad \mathcal{B}_5 = (\mathbf{e}_y + \mathbf{e}_z, \mathbf{e}_z - \mathbf{e}_y, \mathbf{e}_x) \qquad \mathcal{B}_6 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x - \mathbf{e}_y, \mathbf{e}_z)$$

(2) Using your answer to part (1), or otherwise, determine which of the following linear operators preserve the orientation and which change the orientation.

$$\begin{array}{llll}
P_1 : \mathbb{E}^3 \to \mathbb{E}^3 & P_1(\mathbf{e}_x) = \mathbf{e}_y, & P_1(\mathbf{e}_y) = \mathbf{e}_z, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\
P_2 : \mathbb{E}^3 \to \mathbb{E}^3 & P_2(\mathbf{e}_x) = \mathbf{e}_y, & P_2(\mathbf{e}_y) = \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\
P_3 : \mathbb{E}^3 \to \mathbb{E}^3 & P_3(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_3(\mathbf{e}_y) = \mathbf{e}_y - \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\
P_4 : \mathbb{E}^3 \to \mathbb{E}^3 & P_4(\mathbf{e}_x) = \mathbf{e}_y + \mathbf{e}_z, & P_4(\mathbf{e}_y) = \mathbf{e}_z - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\
P_5 : \mathbb{E}^3 \to \mathbb{E}^3 & P_5(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_5(\mathbf{e}_y) = \mathbf{e}_x - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_z
\end{array}$$

Solution. (1) ${}_{\mathcal{B}_1}T_{\mathcal{B}_2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. This has determinant 1 and so $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same orientation.

${}_{\mathcal{B}_1}T_{\mathcal{B}_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This has determinant -1 and so $\mathcal{B}_1$ and $\mathcal{B}_3$ have opposite orientation.

${}_{\mathcal{B}_1}T_{\mathcal{B}_4} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This has determinant 2 and so $\mathcal{B}_1$ and $\mathcal{B}_4$ have the same orientation.

For the remaining transition matrices we can calculate them in the same way or we can note that

$$ {}_{\mathcal{B}_1}T_{\mathcal{B}_4} = {}_{\mathcal{B}_2}T_{\mathcal{B}_5} = {}_{\mathcal{B}_3}T_{\mathcal{B}_6}.$$

Therefore $\mathcal{B}_2$ and $\mathcal{B}_5$ have the same orientation as each other and similarly, $\mathcal{B}_3$ and $\mathcal{B}_6$ share an orientation.

Putting this together we find that the equivalence classes for these bases are

$$\{\,\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4, \mathcal{B}_5\,\} \qquad \text{and} \qquad \{\,\mathcal{B}_3, \mathcal{B}_6\,\}.$$

(2) Notice the P_1 maps basis $\mathcal{B}_1$ to basis $\mathcal{B}_2$, thus the matrix of the linear operator $[P_1]_{\mathcal{B}_1}$ is the same as the transition matrix ${}_{\mathcal{B}_1}T_{\mathcal{B}_2}$. Using the determinant from part (1) we see that P_1 preserves orientation.

Similarly, the matrix of each linear operator P_i in the basis $\mathcal{B}_1$ is the same as the transition matrix ${}_{\mathcal{B}_1}T_{\mathcal{B}_{i+1}}$. Again using the determinants from part (1) we see that P_2 and P_5 change the orientation, whilst P_1, P_3 and P_4 preserve the orientation.

□

Question 2 Consider the matrices

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$$

For each matrix:

(1) find the eigenvalues λ by finding the roots of the polynomial $\det(M - \lambda I)$,

(2) calculate the corresponding eigenvectors.

Solution. The eigenvalues of A are the roots of the polynomial

$$\begin{aligned}\det(A-\lambda I_2) &= \det\begin{bmatrix}5-\lambda & 1\\ 4 & 5-\lambda\end{bmatrix}\\ &= (25-10\lambda+\lambda^2)-4 = \lambda^2-10\lambda+21\\ &= (\lambda-7)(\lambda-3)\end{aligned}$$

The eigenvalues are the values that this polynomial is zero, therefore $\lambda = 3, 7$.

Consider $\lambda = 3$, the corresponding eigenvectors are the solutions to the equation

$$\begin{bmatrix}5 & 1\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 3\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad\Leftrightarrow\quad \begin{aligned}2x_1+x_2&=0\\ 4x_1+2x_2&=0\end{aligned}.$$

We see $[1,-2]^\intercal$ is a solution to this equation, and so the set of eigenvectors with eigenvalue 3 is

$$\operatorname{span}\left(\begin{bmatrix}1\\ -2\end{bmatrix}\right) = \left\{\begin{bmatrix}\alpha\\ -2\alpha\end{bmatrix} \mid \alpha\in\mathbb{R}\right\}.$$

Similarly, the eigenvectors with eigenvalue 7 are the solutions to

$$\begin{aligned}-2x_1+x_2&=0\\ 4x_1+-2x_2&=0\end{aligned} \quad\Rightarrow\quad \begin{bmatrix}x_1\\ x_2\end{bmatrix}\in\operatorname{span}\left(\begin{bmatrix}1\\ 2\end{bmatrix}\right)$$

The eigenvalues of B are the roots of the polynomial

$$\begin{aligned}\det(B-\lambda I_3) &= \det\begin{bmatrix}2-\lambda & 0 & 0\\ -1 & 3-\lambda & 3\\ 6 & -6 & -6-\lambda\end{bmatrix}\\ &= (2-\lambda)\det\begin{bmatrix}3-\lambda & 3\\ -6 & -6-\lambda\end{bmatrix}\\ &= (2-\lambda)(\lambda^2+3\lambda)\\ &= -\lambda(\lambda-2)(\lambda+3)\end{aligned}$$

The eigenvalues are the values that this polynomial is zero, therefore $\lambda = 0, 2, -3$.

The computations for finding the corresponding eigenvectors are identical to the previous case. We find that $\mathbf{x}$ is an eigenvector with eigenvalue λ if:

$$\begin{aligned}\lambda=0 &\Rightarrow \mathbf{x}\in\operatorname{span}\left([0,1,-1]^\intercal\right)\\ \lambda=2 &\Rightarrow \mathbf{x}\in\operatorname{span}\left([1,1,0]^\intercal\right)\\ \lambda=-3 &\Rightarrow \mathbf{x}\in\operatorname{span}\left([0,1,-2]^\intercal\right)\end{aligned}$$

□

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Recall that the reflection operator R in the line spanned by $\mathbf{e}_1$ has matrix

$$[R]_{\mathcal{B}} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Let $\mathbf{x}_\varphi = P_\varphi(\mathbf{e}_1)$ be the vector obtained by rotating $\mathbf{e}_1$ by φ. Let R_φ be the linear operator that is the reflection operator in the line spanned by $\mathbf{x}_\varphi$.

(1) Write R_φ as a composition of rotation operators and the reflection operator R.

(2) Show that $R_\varphi = Q_{2\varphi}$.

Solution. (1) We can perform the operator R_φ by

- rotating the line spanned by $\mathbf{x}_\varphi$ back to the line spanned by $\mathbf{e}_1$,
- performing the reflection R,
- rotating the line back to $\mathbf{x}_\varphi$.

Putting these operators together, we get

$$R_\varphi = P_\varphi \circ R \circ P_{-\varphi}.$$

(2) By writing $[R_\varphi]_\mathcal{B}$, we see that

$$\begin{aligned}
[R_\varphi]_\mathcal{B} &= [P_\varphi]_\mathcal{B}[R]_\mathcal{B}[P_{-\varphi}]_\mathcal{B} \\
&= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \\
&= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\varphi - \sin^2\varphi & 2\sin\varphi\cos\varphi \\ 2\sin\varphi\cos\varphi & -\cos^2\varphi + \sin^2\varphi \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{bmatrix} = [Q_{2\varphi}]_\mathcal{B}
\end{aligned}$$

therefore $R_\varphi = Q_{2\varphi}$.

□

2 Extra Exercises

Question 4 Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{E}^2$, recall that P_φ is the operator that rotates by φ. Show the following identities for matrices of rotation operators in $\mathbb{E}^2$:

(1) $[P_{-\varphi}]_\mathcal{B} = ([P_\varphi]_\mathcal{B})^\top = ([P_\varphi]_\mathcal{B})^{-1}$

(2) $[P_\varphi]_\mathcal{B}[P_\theta]_\mathcal{B} = [P_{\varphi+\theta}]_\mathcal{B}$

Solution. Recall that

$$[P_\varphi]_\mathcal{B} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

(1) We first note that as $[P_\varphi]_\mathcal{B}$ is orthogonal, the transpose and inverse matrix are equal:

$$\begin{aligned} ([P_\varphi]_\mathcal{B})^{-1} &= ([P_\varphi]_\mathcal{B})^{-1} \cdot I_2 \\ &= ([P_\varphi]_\mathcal{B})^{-1} \left([P_\varphi]_\mathcal{B}([P_\varphi]_\mathcal{B})^\top\right) \\ &= \left(([P_\varphi]_\mathcal{B})^{-1}[P_\varphi]_\mathcal{B}\right)([P_\varphi]_\mathcal{B})^\top \\ &= ([P_\varphi]_\mathcal{B})^\top \end{aligned}$$

Now using the fact $\cos(-\varphi) = \cos(\varphi)$ and $\sin(-\varphi) = -\sin(\varphi)$, we have

$$[P_{-\varphi}]_\mathcal{B} = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = ([P_\varphi]_\mathcal{B})^\top.$$

(2) This can be solved via matrix multiplication and trigonometric identities.

$$\begin{aligned} [P_\varphi]_\mathcal{B}[P_\theta]_\mathcal{B} &= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta & \sin\varphi\cos\theta - \cos\varphi\sin\theta \\ \sin\varphi\cos\theta + \cos\varphi\sin\theta & \cos\varphi\cos\theta - \sin\varphi\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{bmatrix} \\ &= [P_{\varphi+\theta}]_\mathcal{B} \end{aligned}$$

The geometric interpretation of this is the composition of two rotations by φ and θ is the same as rotating by the angle $\varphi + \theta$.

□

Question 5 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Consider the linear operators P_1, P_2 on $\mathbb{E}^2$ defined by

$$\begin{aligned} P_1(\mathbf{e}_1) &= \mathbf{e}_1 & \qquad P_2(\mathbf{e}_1) &= \mathbf{e}_1 - \mathbf{e}_2 \\ P_1(\mathbf{e}_2) &= \mathbf{e}_1 + \mathbf{e}_2 & \qquad P_2(\mathbf{e}_2) &= \mathbf{e}_2 \end{aligned}$$

(1) Write down the matrices for P_1, P_2. Which of them are orthogonal operators?

(2) Find all linear operators of the form $P = aP_1 + bP_2$ that are orthogonal (where $a, b \in \mathbb{R}$).

(3) For each orthogonal P, write it as either a rotation operator P_φ or a reflection operator Q_φ.

Solution. (1)

$$[P_1]_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad [P_2]_\mathcal{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Neither operator is orthogonal as their matrices are not orthogonal, *i.e.*, they do not satisfy $M^\top M = I_2$.

(2) Consider the matrix for P

$$[P]_\mathcal{B} = a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a \\ -b & a+b \end{bmatrix}.$$

P is orthogonal if and only if $[P]_{\mathcal{B}}$ is orthogonal, i.e.

$$\begin{bmatrix} a+b & a \\ -b & a+b \end{bmatrix} \begin{bmatrix} a+b & -b \\ a & a+b \end{bmatrix} = \begin{bmatrix} (a+b)^2+a^2 & (a-b)(a+b) \\ (a-b)(a+b) & (a+b)^2+b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $(a-b)(a+b) = 0$, then we have two cases $a = b$ and $a = -b$.
If $a = -b$, then

$$(a+b)^2 + a^2 = a^2 = 1 \Rightarrow a = \pm 1 \Rightarrow P = \begin{cases} P_1 - P_2 & a = +1 \\ -P_1 + P_2 & a = -1 \end{cases}.$$

If $a = b$, then

$$(a+b)^2 + a^2 = 5a^2 = 1 \Rightarrow a = \pm\frac{1}{\sqrt{5}} \Rightarrow P = \begin{cases} \frac{1}{\sqrt{5}}P_1 + \frac{1}{\sqrt{5}}P_2 & a = +\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}}P_1 - \frac{1}{\sqrt{5}}P_2 & a = -\frac{1}{\sqrt{5}} \end{cases}.$$

(3) Let $P = P_1 - P_2$, then

$$[P]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\det(P) = -1$ and so P is of the form Q_φ. Explicitly

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} \quad \Rightarrow \quad \varphi = \frac{\pi}{2}.$$

Similarly, $P_2 - P_1 = Q_{-\frac{\pi}{2}}$.
Let $P = \frac{1}{\sqrt{5}}P_1 + \frac{1}{\sqrt{5}}P_2$, then

$$[P]_{\mathcal{B}} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Note that $\det(P) = 1$ and so P is of the form P_φ. Explicitly

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \Rightarrow \quad \varphi = \arctan\left(\frac{-1}{2}\right) \simeq -26.6^\circ$$

A similar calculation shows that $-\frac{1}{\sqrt{5}}P_1 - \frac{1}{\sqrt{5}}P_2 = P_{\pi+\varphi}$ (although this requires a little care as arctan has period π).

□