Exercises 2

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

A lot of these questions are key calculations; as a result we've made a lot of them available but do not feel you have to attempt them all, especially if you are comfortable with the calculation.

Question 1 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) orthonormal basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

(1) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_3$

(2) $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{f}_3 = \mathbf{e}_3$

(3) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{f}_3 = \mathbf{e}_3$

(4) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ orthogonal?

[*Hint: You can use the transition matrices from Exercises 1, Question 3*]

Solution. (1) From Exercises 1, Question 3, we know that

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -1 $$

As $({}_{\mathcal{B}}T_{\mathcal{C}})^\top({}_{\mathcal{B}}T_{\mathcal{C}}) = I_3$ holds, and the basis is orthogonal.

(2) From Exercises 1, Question 3, we know that

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0 $$

As the determinant is not equal to ± 1, it immediately cannot be orthogonal.

(3) From Exercises 1, Question 3, we know that

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0 $$

As the determinant is not equal to ± 1, it immediately cannot be orthogonal.

(4) From Exercises 1, Question 3, we know that

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -\lambda $$

$\mathcal{C}$ is not orthogonal - if we calculate $({}_{\mathcal{B}}T_{\mathcal{C}})^\top({}_{\mathcal{B}}T_{\mathcal{C}})$, we see

$$ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & 1 & \lambda \\ 1 & 2 & \lambda \\ \lambda & \lambda & \lambda^2 \end{bmatrix} \neq I_3. $$

One could also notice this by considering the length of $\mathbf{f}_3$:

$$ \|\mathbf{f}_3\| = \sqrt{2 + \lambda^2} > 1, $$

therefore $\mathcal{C}$ cannot be orthonormal as $\mathbf{f}_3$ is not unit length.

□

Question 2 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for $\mathbb{R}^2$. Write down the matrix for the following linear operators in the basis $\mathcal{B}$.

(1)
$$\begin{aligned} P_1\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x \\ \mathbf{e}_y &\mapsto 3\mathbf{e}_y \end{aligned}$$

(2)
$$\begin{aligned} P_2\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x - \mathbf{e}_y \end{aligned}$$

(3)
$$\begin{aligned} P_3\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto -4\mathbf{e}_x - 2\mathbf{e}_y \end{aligned}$$

(4)
$$\begin{aligned} P_4\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x + 2\mathbf{e}_y \end{aligned}$$

Solution. (1) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ (4) $\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

□

Question 3 Let $\mathcal{B}, \mathcal{C}$ be (ordered) bases of a vector space V. Show that the transition matrix ${}_{\mathcal{C}}T_{\mathcal{B}}$ from $\mathcal{C}$ to $\mathcal{B}$ is the inverse matrix of the transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ from $\mathcal{B}$ to $\mathcal{C}$ i.e.

$${}_{\mathcal{C}}T_{\mathcal{B}} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

Solution. We can solve this very quickly by using the transitivity property of transition matrices from Lemma 1.41 in the notes. In particular, we have that ${}_{\mathcal{C}}T_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}} = {}_{\mathcal{C}}T_{\mathcal{C}}$. The transition matrix from $\mathcal{C}$ to $\mathcal{C}$ is just the identity matrix, therefore

$${}_{\mathcal{C}}T_{\mathcal{B}} = {}_{\mathcal{C}}T_{\mathcal{B}} I_n = {}_{\mathcal{C}}T_{\mathcal{B}}\,{}_{\mathcal{B}}T_{\mathcal{C}}\,({}_{\mathcal{B}}T_{\mathcal{C}})^{-1} = I_n({}_{\mathcal{B}}T_{\mathcal{C}})^{-1} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

□

Question 4 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an orthonormal basis for $\mathbb{E}^3$. Calculate the determinant and trace for each of the following linear operators:

(1)
$$\begin{aligned} P_1\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x + \mathbf{e}_z \end{aligned}$$

(2)
$$\begin{aligned} P_2\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x - \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y - \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x - \mathbf{e}_z \end{aligned}$$

(3)
$$\begin{aligned} P_3\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x \\ \mathbf{e}_y &\mapsto -\mathbf{e}_z \\ \mathbf{e}_z &\mapsto -\mathbf{e}_y \end{aligned}$$

(4)
$$\begin{aligned} P_4\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \sqrt{2}\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_y &\mapsto -\sqrt{2}(\mathbf{e}_y + \mathbf{e}_z) \\ \mathbf{e}_z &\mapsto \sqrt{2}\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z \end{aligned}$$

Solution. (1)

$$[P_1]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathrm{Tr}(P) = 3 \qquad \det(P) = \det\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] + \det\left[\begin{smallmatrix} 1 & 1 \\ \mathbf{0} & 1 \end{smallmatrix}\right] = 2$$

(2)

$$[P_2]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \qquad \mathrm{Tr}(P) = 1$$

For the determinant we can notice that the sum of the rows is zero, and hence the matrix is degenerate and $\det P = 0$. Alternatively we can calculate the determinant directly: $\det(P) = \det\left[\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right] + \det\left[\begin{smallmatrix} -1 & 1 \\ \mathbf{0} & -1 \end{smallmatrix}\right] = 0$

(3)

$$[P_3]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \mathrm{Tr}(P) = 1 \qquad \det(P) = \det\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right] = -1$$

(4)

$$[P_4]_{\mathcal{B}} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & -\sqrt{2} & 1 \\ 1 & -\sqrt{2} & -1 \end{bmatrix} \qquad \operatorname{Tr}(P) = -1$$

$$\det(P) = \sqrt{2}\det\begin{bmatrix} -\sqrt{2} & 1 \\ -\sqrt{2} & -1 \end{bmatrix} + \sqrt{2}\det\begin{bmatrix} -1 & -\sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} = \sqrt{2}(2\sqrt{2}) + \sqrt{2}(2\sqrt{2}) = 8$$

□

2 Extra Exercises

Question 5 Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be vectors in $\mathbb{E}^3$ such that $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, and $\mathbf{c}$ has length $\sqrt{3}$ and forms the angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with $\mathbf{a}$ and $\mathbf{b}$.

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ forms an orthonormal basis for $\mathbb{E}^3$.

Solution. $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, we immediately have

$$\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = 1, \langle \mathbf{a}, \mathbf{b} \rangle = 0.$$

We also know that as $\|\mathbf{c}\| = \sqrt{\langle \mathbf{c}, \mathbf{c} \rangle} = \sqrt{3}$, we see that $\langle \mathbf{c}, \mathbf{c} \rangle = 3$. Furthermore, we can calculate $\langle \mathbf{a}, \mathbf{c} \rangle$ via

$$\langle \mathbf{a}, \mathbf{c} \rangle = \|\mathbf{a}\|\|\mathbf{c}\| \cos \varphi = 1 \cdot \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1.$$

We also get $\langle \mathbf{b}, \mathbf{c} \rangle = 1$ via the same calculation.

To show $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ form an orthonormal basis, we need to show they have unit length and are pairwise orthogonal. We know already $\mathbf{a}, \mathbf{b}$ are orthogonal and have unit length. To see $\mathbf{c} - \mathbf{a} - \mathbf{b}$ is orthogonal to $\mathbf{a}$ (and by a very similar calculation $\mathbf{b}$), we get

$$\langle \mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{a} \rangle = \langle \mathbf{c}, \mathbf{a} \rangle - \langle \mathbf{a}, \mathbf{a} \rangle - \langle \mathbf{b}, \mathbf{a} \rangle = 1 - 0 - 1 = 0.$$

Finally, to check $\mathbf{c} - \mathbf{a} - \mathbf{b}$ has unit length, we get

$$\begin{aligned}\langle \mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b} \rangle &= \langle \mathbf{c}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{a} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle + 2\langle \mathbf{a}, \mathbf{b} \rangle - 2\langle \mathbf{a}, \mathbf{c} \rangle - 2\langle \mathbf{b}, \mathbf{c} \rangle \\ &= 3 + 1 + 1 + 2 \cdot 0 - 2 \cdot 1 - 2 \cdot 1 = 1,\end{aligned}$$

therefore $\|\mathbf{c}\| = \sqrt{\langle \mathbf{c}, \mathbf{c} \rangle} = 1$. □

Question 6 Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be vectors of $\mathbb{E}^3$ such that each has unit length and they are pairwise orthogonal with each other.

(1) Show that they are linearly independent.

(2) Show that $\mathcal{C}$ forms a basis for $\mathbb{E}^3$.

Solution. (1) We want to show $\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 + \lambda_3 \mathbf{f}_3 = \mathbf{0}$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Take the inner product of $\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 + \lambda_3 \mathbf{f}_3$ with $\mathbf{f}_1$:

$$\langle \lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 + \lambda_3 \mathbf{f}_3, \mathbf{f}_1 \rangle = \lambda_1 \langle \mathbf{f}_1, \mathbf{f}_1 \rangle + \lambda_2 \langle \mathbf{f}_2, \mathbf{f}_2 \rangle + \lambda_3 \langle \mathbf{f}_3, \mathbf{f}_1 \rangle = \lambda_1 \cdot 1 + \lambda_2 \cdot 0 + \lambda_3 \cdot 0 = 0$$

implying $\lambda_1 = 0$. Taking the inner product with $\mathbf{f}_2$ and $\mathbf{f}_3$ gives $\lambda_2 = \lambda_3 = 0$, therefore $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ are linearly independent.

(2) Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis for $\mathbb{E}^3$; we can write $\mathcal{C}$ in terms of $\mathcal{B}$:

$$\begin{aligned}\mathbf{f}_1 &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 \\ \mathbf{f}_2 &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 \\ \mathbf{f}_3 &= a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3\end{aligned} \quad \Leftrightarrow \quad {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

By the notes, we know that $\mathcal{C}$ forms a basis if and only if $\operatorname{rk}({}_{\mathcal{B}}T_{\mathcal{C}}) = n$. By the rank-nullity theorem:

$$\operatorname{rk}({}_{\mathcal{B}}T_{\mathcal{C}}) + \operatorname{null}({}_{\mathcal{B}}T_{\mathcal{C}}) = n \;\Leftrightarrow\; \operatorname{null}({}_{\mathcal{B}}T_{\mathcal{C}}) = \dim(\ker({}_{\mathcal{B}}T_{\mathcal{C}})) = 0,$$

therefore $\mathcal{C}$ forms a basis if and only if $\ker({}_{\mathcal{B}}T_{\mathcal{C}}) = \{\mathbf{0}\}$.

Let $\boldsymbol{\lambda} = [\lambda_e, \lambda_f, \lambda_g]^\mathsf{T} \in \ker({}_{\mathcal{B}}T_{\mathcal{C}})$ be an element of the kernel of ${}_{\mathcal{B}}T_{\mathcal{C}}$.

$$\begin{aligned}\mathbf{0} = {}_{\mathcal{B}}T_{\mathcal{C}}\boldsymbol{\lambda} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 a_{11} + \lambda_2 a_{21} + \lambda_3 a_{31} \\ \lambda_1 a_{12} + \lambda_2 a_{22} + \lambda_3 a_{32} \\ \lambda_1 a_{13} + \lambda_2 a_{23} + \lambda_3 a_{33} \end{bmatrix} \\ &= \lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 + \lambda_3 \mathbf{f}_3.\end{aligned}$$

As $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are linearly independent, we must have $\lambda_1 = \lambda_2 = \lambda_3$, implying that the only element of $\ker({}_{\mathcal{B}}T_{\mathcal{C}})$ is the zero vector. This implies that $\mathcal{C}$ is a basis.

□

Question 7 Let $P\colon V \to V$ be a linear operator acting on the vector space V and let $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis for V. Prove that for any $\mathbf{v} \in V$, the column vector representing the image of $\mathbf{v}$ in the $\mathcal{B}$ basis, $[P(\mathbf{v})]_{\mathcal{B}}$, is given by the matrix multiplication: $[P]_{\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}$.

You may use without proof the fact that matrix multiplication is linear: that is, for a matrix M, column vectors v, w and scalars λ, μ the following holds:

$$M(\lambda v + \mu w) = \lambda M v + \mu M w$$

Solution. Fix an arbitrary vector $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$. We have

$$P(\mathbf{v}) = P\left(\sum_{i=1}^{n} v_i \mathbf{e}_i\right) = \sum_{i=1}^{n} v_i P(\mathbf{e}_i) \qquad \text{(linearity of } P\text{)}$$

therefore by linearity of the notation $[-]_{\mathcal{B}}$ we also have

$$\begin{aligned}
[P(\mathbf{v})]_{\mathcal{B}} &= \sum_{i=1}^{n} v_i [P(\mathbf{e}_i)]_{\mathcal{B}} \\
&= \sum_{i=1}^{n} v_i [P]_{\mathcal{B}}\,[\mathbf{e}_i]_{\mathcal{B}} && ([P(\mathbf{e}_i)]_{\mathcal{B}} \text{ is the } i^{\text{th}} \text{ column of } [P]_{\mathcal{B}}) \\
&= [P]_{\mathcal{B}} \left(\sum_{i=1}^{n} v_i [\mathbf{e}_i]_{\mathcal{B}}\right) && \text{(linearity of matrix multiplication)} \\
&= [P]_{\mathcal{B}} \left[\left(\sum_{i=1}^{n} v_i \mathbf{e}_i\right)\right]_{\mathcal{B}} && \text{(linearity of } [-]_{\mathcal{B}}) \\
&= [P]_{\mathcal{B}}\,[\mathbf{v}]_{\mathcal{B}}
\end{aligned}$$

□

Question 8 Let P be a linear operator acting on 2-dimensional vector space V and let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for V, such that the matrix of P in the basis $\mathcal{B}$ is given by

$$[P]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}.$$

Show that there is a basis $\mathcal{C} = (\mathbf{e}_u, \mathbf{e}_v)$ such that the linear operator in the basis $\mathcal{C}$ is

$$[P]_{\mathcal{C}} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$

Hint: *The second matrix means that $P(\mathbf{e}_u) = 4\mathbf{e}_u$ (it is an eigenvector). Use this fact to equate $[P(\mathbf{e}_u)]_{\mathcal{B}}$ with $[4\mathbf{e}_u]_{\mathcal{B}}$.*

Solution. The second matrix tells us that $P(\mathbf{e}_u) = 4\mathbf{e}_u$. We can write $\mathbf{e}_u$ in the $\mathcal{B}$ basis: $\mathbf{e}_u = \lambda \mathbf{e}_x + \mu \mathbf{e}_y$. Then

$$\begin{bmatrix} 4\lambda \\ 4\mu \end{bmatrix} = [4\mathbf{e}_u]_{\mathcal{B}} = [P(\mathbf{e}_u)]_{\mathcal{B}} = [P]_{\mathcal{B}}\,[\mathbf{e}_u]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 5\lambda - \mu \\ 2\lambda + 2\mu \end{bmatrix}$$

so $\lambda = \mu$.

Similarly we see that $P(\mathbf{e}_v) = 3\mathbf{e}_v$, and if $\mathbf{e}_v = \alpha \mathbf{e}_x + \beta \mathbf{e}_y$ we have

$$\begin{bmatrix} 3\alpha \\ 3\beta \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5\alpha - \beta \\ 2\alpha + 2\beta \end{bmatrix}$$

so $2\alpha = \beta$. We have two degrees of freedom to choose the basis, one such choice is $\mathcal{C} = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x + 2\mathbf{e}_y)$.

The transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, with inverse $\left[\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ and we may check that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ \mathbf{0} & 3 \end{bmatrix}$$

□

Question 9 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be an orthonormal basis for 2-dimensional Euclidean space $\mathbb{E}^2$.

(1) Consider the following alternative bases:

(a) $\mathcal{C}_{1a} = (\mathbf{e}_x, -\mathbf{e}_y)$ (b) $\mathcal{C}_{1b} = (\frac{\mathbf{e}_x+\mathbf{e}_y}{\sqrt{2}}, \frac{\mathbf{e}_y-\mathbf{e}_x}{\sqrt{2}})$

In each case write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}_\bullet$ and calculate the transition matrix from $\mathcal{C}_\bullet$ to $\mathcal{B}$.

(2) For each of the linear operators $P_{(1)}$ and $P_{(2)}$ from question Question 2 calculate the matrix for the linear operator in each $\mathcal{C}$ basis above.

Solution. (1) Using the fact that ${}_{\mathcal{C}}T_{\mathcal{B}} = {}_{\mathcal{B}}T_{\mathcal{C}}{}^{-1}$

(a)
$$ {}_{\mathcal{B}}T_{\mathcal{C}_{1a}} = \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix} \qquad {}_{\mathcal{C}_{1a}}T_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix}$$

(b)
$$ {}_{\mathcal{B}}T_{\mathcal{C}_{1b}} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad {}_{\mathcal{C}_{1b}}T_{\mathcal{B}} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(2) (a)

$$\begin{aligned} \left[P_{(1)}\right]_{\mathcal{C}_{(1a)}} &= {}_{\mathcal{C}_{(1a)}}T_{\mathcal{B}} \left[P_{(1)}\right]_{\mathcal{B}} {}_{\mathcal{B}}T_{\mathcal{C}_{(1a)}} \\ &= \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned} \qquad \begin{aligned} \left[P_{(2)}\right]_{\mathcal{C}_{(1a)}} &= {}_{\mathcal{C}_{(1a)}}T_{\mathcal{B}} \left[P_{(2)}\right]_{\mathcal{B}} {}_{\mathcal{B}}T_{\mathcal{C}_{(1a)}} \\ &= \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{0} & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \left[P_{(1)}\right]_{\mathcal{C}_{(1b)}} &= {}_{\mathcal{C}_{(1b)}}T_{\mathcal{B}} \left[P_{(1)}\right]_{\mathcal{B}} {}_{\mathcal{B}}T_{\mathcal{C}_{(1b)}} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix} \end{aligned} \qquad \begin{aligned} \left[P_{(2)}\right]_{\mathcal{C}_{(1b)}} &= {}_{\mathcal{C}_{(1b)}}T_{\mathcal{B}} \left[P_{(2)}\right]_{\mathcal{B}} {}_{\mathcal{B}}T_{\mathcal{C}_{(1b)}} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

□

Question 10 For each linear operator in Question 4:

(1) Determine if the operator is orthogonal or not.

Note: *It may be useful to recall a fact we learnt about the determinant of an orthogonal operator.*

(2) For $\lambda \in \mathbb{R}$ define the scaled linear operator $S\colon \mathbb{E}^3 \to \mathbb{E}^3$, by $S(\mathbf{v}) = \lambda P(\mathbf{v})$. Determine the values of λ (if any exist) such that S is an orthogonal operator.

Solution. (1) The determinant of an orthogonal operator is either 1 or −1, so we can see immediately that $P_{(1)}$, $P_{(2)}$ and $P_{(4)}$ are **not** orthogonal operators.

For $P_{(3)}$ we can check if the matrix of the linear operator in the orthonormal basis $\mathcal{B}$, is an orthogonal matrix.

$$\left[P_{(3)}\right]_{\mathcal{B}} \left[P_{(3)}\right]_{\mathcal{B}}{}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = I_3.$$

Since the product gives the identity, $P_{(3)}$ is an orthogonal operator.

(2) If $S_{(1)}$ is orthogonal then the inner product $\left\langle S_{(1)}(\mathbf{e}_x), S_{(1)}(\mathbf{e}_y)\right\rangle = 0$, but

$$\begin{aligned} \left\langle S_{(1)}(\mathbf{e}_x), S_{(1)}(\mathbf{e}_y)\right\rangle &= \left\langle \lambda P_{(1)}(\mathbf{e}_x), \lambda P_{(1)}(\mathbf{e}_y)\right\rangle \\ &= \lambda^2 \langle \mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y + \mathbf{e}_z\rangle = \lambda^2. \end{aligned}$$

Thus if $\left\langle S_{(1)}(\mathbf{e}_x), S_{(1)}(\mathbf{e}_y)\right\rangle = 0$, we would necessarily require $\lambda = 0$. Since the zero linear operator is never orthogonal we see that $S_{(1)}$ is not orthogonal for any choice of λ.

The operator $P_{(2)}$ is degenerate, thus the scaled version $S_{(2)}$ is also degenerate. It is therefore not orthogonal for any $\lambda \in \mathbb{R}$.

The operator $P_{(3)}$ is orthogonal. It is easy to see that $-P_{(3)}$ is also orthogonal, but for any other choice of λ the determinant would not be ± 1. Thus $S_{(3)}$ is orthogonal for $\lambda \in \{ 1, -1 \}$.

The operator $P_{(4)}$ has determinant 8 and the determinant of $S_{(4)} = \lambda^3 \det P_{(4)}$, thus the only possible choices for λ are $\pm 1/2$. Setting $\lambda = 1/2$ we see that

$$\left[S_{(4)}\right]_{\mathcal{B}} \left[S_{(4)}\right]_{\mathcal{B}}^{\mathsf{T}} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & -1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \end{bmatrix} = I_3.$$

Similarly for $\lambda = -1/2$, so $S_{(4)}$ is orthogonal for $\lambda \in \{ 1/2, -1/2 \}$.

$\square$