Exercises 1

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

(1) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$.

(2) Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis for $\mathbb{R}^2$.

(3) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is *not* a basis for $\mathbb{R}^2$.

Solution. For each pair of vectors, we have to show they are linearly independent and span $\mathbb{R}^2$.

(1)

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 = \mathbf{0} \Rightarrow \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0$$

Therefore they are linearly independent. For any arbitrary vector $\mathbf{x} = (x_1, x_2)^\mathsf{T}$ of $\mathbb{R}^2$, we can write it as

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

therefore $\{\mathbf{e}_1, \mathbf{e}_2\}$ span $\mathbb{R}^2$.

(2)

$$\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} = \mathbf{0} \Rightarrow \lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 + 3\lambda_2 \\ 3\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second coordinate gives $\lambda_1 = 0$ and therefore $\lambda_2 = 0$, and so they are linearly independent. For any arbitrary vector $\mathbf{x} = (x_1, x_2)^\mathsf{T}$ of $\mathbb{R}^2$, we can write it as the linear combination

$$\lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\Rightarrow \lambda_1 = \frac{x_2}{3}, \ \lambda_2 = \frac{x_1 - 2\lambda_1}{3} = \frac{x_1}{3} - \frac{2x_2}{9},$$

therefore $\{\mathbf{a}, \mathbf{b}\}$ span $\mathbb{R}^2$.

(3)

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{b} = \mathbf{0} \Rightarrow \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 + 3\lambda_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This holds for $\lambda_1 = 3, \lambda_2 = -1$, therefore we have a linear dependence with non-zero coefficients and so they do not form a basis.

□

Question 2 State whether each of the following maps $\langle -, - \rangle$ define an inner product on $\mathbb{R}^3$. [*where* $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}, \mathbf{y} = (y_1, y_2, y_3)^\mathsf{T}$.]

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$

(2) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$

(3) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + x_2 y_1 + x_3 y_3$

Solution. (1) $\langle -, - \rangle$ is not an inner product as it is not positive definite. To see this, $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2$, which is zero for certain non-zero vectors, such as $(0, 0, -1)^\mathsf{T}$.

(2) $\langle -, - \rangle$ is an inner product. We need to check symmetry, linearity and that it is positive definite.

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3 = y_1 x_1 + 3y_2 x_2 + 5y_3 x_3 = \langle \mathbf{y}, \mathbf{x} \rangle,$$

and so symmetry holds.

$$\begin{aligned}\langle \lambda\mathbf{x} + \mu\mathbf{y},\, \mathbf{z}\rangle &= (\lambda x_1 + \mu y_1)z_1 + 3(\lambda x_2 + \mu y_2)z_2 + 5(\lambda x_3 + \mu y_3)z_3 \\ &= \lambda(x_1z_1 + 3x_2z_2 + 5x_3z_3) + \mu(y_1z_1 + 3y_2z_2 + 5y_3z_3) \\ &= \lambda\langle \mathbf{x},\, \mathbf{z}\rangle + \mu\langle \mathbf{y},\, \mathbf{z}\rangle\end{aligned}$$

and so linearity holds. Finally

$$\langle \mathbf{x},\, \mathbf{x}\rangle = x_1^2 + 3x_2^2 + 5x_3^2 \geq 0$$

is greater than or equal to zero for elements $\mathbf{x} \in \mathbb{R}^3$. Furthermore, there is equality if and only if $x_1^2 = 3x_2^2 = 5x_3^2 = 0$, which only occurs for $\mathbf{x} = \mathbf{0}$. Therefore positive-definite also holds.

(3) $\langle -, -\rangle$ does not define an inner product as it does not satisfy positive definiteness:

$$\langle \mathbf{x},\, \mathbf{x}\rangle = 2x_1x_2 + x_3^2$$

takes negative values at $\mathbf{x} = (-1, 1, 0)^\mathsf{T}$.

□

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

(1) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_3$

(2) $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{f}_3 = \mathbf{e}_3$

(3) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{f}_3 = \mathbf{e}_3$

(4) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ a basis?

Solution. (1)

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -1$$

The transition matrix has non-zero determinant, and so $\mathcal{C}$ is a basis. Moreover, we see $({}_{\mathcal{B}}T_{\mathcal{C}})^\mathsf{T}({}_{\mathcal{B}}T_{\mathcal{C}}) = I_3$ holds, and so the basis is orthogonal also.

(2)

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0$$

As the transition matrix has determinant zero, $\mathcal{C}$ is not a basis. One could also note that the rank must be ≤ 2 as ${}_{\mathcal{B}}T_{\mathcal{C}}$ has a row of zeros.

(3)

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0$$

As the transition matrix has determinant zero, $\mathcal{C}$ is not a basis. One could also notice the first two columns are proportional, and so there is a linear dependence between $\mathbf{f}_1, \mathbf{f}_2$.

(4)

$${}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -\lambda$$

The transition matrix has nonzero determinant whenever $\lambda \neq 0$, and so $\mathcal{C}$ forms a basis when this occurs.

□

2 Extra exercises

Question 4 Consider the sets of polynomials

$$V = \left\{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\right\}, T = \left\{x^2 + px + q \mid p, q \in \mathbb{R}\right\}$$

with the natural operations of addition and multiplication of polynomials. [*You may assume these operations satisfy commutativity, associativity and distributivity.*]

(1) Which of these are vector spaces (over $\mathbb{R}$), and why?

(2) Show the polynomials $1, x, x^2$ are linearly independent in V.

(3) Calculate the dimension of V.

Solution. (1) T is not a vector space, as the sum of any two polynomials does not belong to T e.g. $x^2 + x^2 = 2x^2 \notin T$.

V is a vector space. Let $f = a_2x^2 + a_1x + a_0, g = b_2x^2 + b_1x + b + 0$, to see the remaining axioms hold:

- (Zero) Setting $a_2 = a_1 = a_0 = 0$ gives $f = 0$ as the zero element.
- (Unity) Clearly $1 \cdot f = f$.
- (Additive inverses) For f, the inverse polynomial is $-f = (-a_2)x^2 + (-a_1)x + (-a_0)$.
- (Additive closure) The sum $f + g = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$ is contained in V.
- (Multiplicative closure) $\lambda \cdot f = (\lambda a_2)x^2 + (\lambda a_1)x + (\lambda a_0)$ is contained in V for any choice of $\lambda \in \mathbb{R}$.

(2) Suppose $1, x, x^2$ satisfy an identity

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 = \mathbf{0}(= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2)$$

for some choice of $c_0, c_1, c_2 \in \mathbb{R}$. Equivalently, the polynomial $P(x) = c_0 + c_1x + c_2x^2$ equals zero for all choices of x. Testing this on the values $x = 0, 1, -1$, we see

$$\begin{aligned} P(0) &= c_0 \Rightarrow c_0 = 0 \\ P(1) &= c_1 + c_2 \Rightarrow c_1 = -c_2 \\ P(-1) &= -c_1 + c_2 \Rightarrow c_1 = c_2 \\ &\Rightarrow c_0 = c_1 = c_2 = 0 \end{aligned}$$

Therefore the only linear combination between $\{1, x, x^2\}$ equal to zero is when all coefficients are zero, and so they are linearly independent.

(3) $\{1, x, x^2\}$ are linearly independent and span V, therefore they form a basis. Therefore $\dim(V) = 3$. □

Question 5 Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be vectors of a vector space V. Show that if at least one of the vectors is equal to $\mathbf{0}$, then they are linearly dependent.

Solution. Without loss of generality, let us say $\mathbf{a}_1 = \mathbf{0}$. We can pick any arbitrary real $\lambda \neq 0$ such that

$$\lambda \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \cdots + 0 \cdot \mathbf{a}_n = \mathbf{0}.$$

As we have found a linear combination where not all coefficients are zero, these vectors are linearly dependent. □