Exercises 11

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Consider the three points

$$\mathbf{A} = [-1 : 1 : 0], \quad \mathbf{B} = [2 : 2 : 4], \quad \mathbf{C} = [3 : 1 : 4].$$

(1) Show these points are collinear,

(2) For each affine chart, find a point at infinity that is also collinear to these three points.

Solution. (1) Checking the determinant of the matrix of points, we see that

$$\det \begin{bmatrix} -1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix} = -1(8-4) - 1(8-12) = 0$$

therefore they are collinear.

(2) A point at infinity on the first affine chart has the form $[0 : v : w]$. Furthermore, we would like it to be collinear, therefore we look at the following matrix and check when its determinant is zero:

$$\det \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & v \\ 0 & 4 & w \end{bmatrix} = -1(2w - 4v) - 2w = 0 \Rightarrow v = w.$$

Therefore the point $[0 : 1 : 1]$ is collinear.

For the second affine chart, we consider the point $[u : 0 : w]$:

$$\det \begin{bmatrix} -1 & 2 & u \\ 1 & 2 & 0 \\ 0 & 4 & w \end{bmatrix} = -1(2w) - 2(w) + u(4) = 0 \Rightarrow u = w.$$

Therefore the point $[1 : 0 : 1]$ is collinear.

For the third affine chart, we note the point $[-1 : 1 : 0]$ is already a point at infinity. Every line on an affine chart has at most one point at infinity, so this is the only point. A similar check to the other parts reveals this.

□

Question 2 For each of the following set of points, check whether they are collinear, and if so calculate their cross-ratio $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$.

(1) $\mathbf{A} = [1 : 0 : -2]$, $\mathbf{B} = [-2 : 1 : -1]$, $\mathbf{C} = [-4 : 1 : 3]$, $\mathbf{D} = [5 : -2 : 0]$

(2) $\mathbf{A} = [0 : 1 : -1]$, $\mathbf{B} = [-1 : 2 : 1]$, $\mathbf{C} = [1 : 1 : -4]$, $\mathbf{D} = [1 : 0 : 2]$

(3) $\mathbf{A} = [1 : 0 : -1]$, $\mathbf{B} = [0 : 1 : 1]$, $\mathbf{C} = [1 : 1 : 0]$, $\mathbf{D} = [1 : 5 : 4]$

Solution. (1) By checking determinants, we can confirm these points are collinear. Using $\mathbb{A}^2_2$, these points have affine coordinates

$$\mathbf{A} = \infty, \mathbf{B} = (-2, -1), \mathbf{C} = (-4, 3), \mathbf{D} = \left(\frac{-5}{2}, 0\right),$$
$$\Rightarrow \quad (\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{\infty}{\infty} \cdot \frac{[1/2, -1]}{[2, -4]} = \frac{1}{4}$$

(2) While $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, $\mathbf{D}$ is not: the matrix formed by $\mathbf{A}, \mathbf{B}, \mathbf{D}$ does not have zero determinant:

$$\det \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix} = 1(2) + (1 - (-2)) = 5.$$

(3) *[An old version of this question had the typo* $\mathbf{C} = [1 : -1 : 0]$*, in which case these are not collinear!]*

By checking determinants, we can confirm these points are collinear. Using $\mathbb{A}^2_1$, these points have affine coordinates

$$\mathbf{A} = (0, -1),\ \mathbf{B} = \infty,\ \mathbf{C} = (1, 0),\ \mathbf{D} = (5, 4)\,,$$
$$\Rightarrow \quad (\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{[-1, -1]}{[-5, -5]} \cdot \frac{\infty}{\infty} = \frac{1}{5}$$

□

Question 3 For each trio of points $\mathbf{A}, \mathbf{B}, \mathbf{C}$, find a fourth point $\mathbf{D}$ such that $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 3$.

(1) $\mathbf{A} = [1 : 3 : 2]$, $\mathbf{B} = [1 : 1 : 1]$, $\mathbf{C} = [1 : -1 : 0]$

(2) $\mathbf{A} = [0 : 1 : 2]$, $\mathbf{B} = [1 : 0 : 2]$, $\mathbf{C} = [-1 : 1 : 0]$

Solution. (1) Using the first affine chart, we get

$$\mathbf{A} = (3, 2),\ \mathbf{B} = (1, 1),\ \mathbf{C} = (-1, 0),\ \mathbf{D} = (v, w)$$
$$\Rightarrow \quad (\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{[4, 2]}{[3 - v, 2 - w]} \cdot \frac{[1 - v, 1 - w]}{[2, 1]} = 2\frac{[1 - v, 1 - w]}{[3 - v, 2 - w]} = 3.$$

Note the ratio must be satisfied in both the v and w coordinate, therefore:

$$\begin{matrix} 2(1 - v) = 3(3 - v) \\ 2(1 - w) = 3(2 - w) \end{matrix} \Rightarrow v = 7, w = 4 \Rightarrow \mathbf{D} = [1 : 7 : 4].$$

[Alternatively, we could notice that the points live on the line $v = 2w - 1$*, therefore we can write* $\mathbf{D}$ *in one unknown as* $(2w - 1, w)$ *and solve directly from the cross-ratio.]*

(2) Using the second affine chart we get

$$\mathbf{A} = (0, 2)\,, \mathbf{B} = \infty,\ \mathbf{C} = (-1, 0), \mathbf{D} = (u, w)$$
$$\Rightarrow \quad (\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{[1, 2]}{[-u, 2 - w]} = 3.$$

Again by satisfying the ratio in each coordinate, we get $\mathbf{D} = [-\frac{1}{3} : 1 : \frac{4}{3}] = [-1 : 3 : 4]$.

[Alternatively, we could notice that the points live on the line $w = 2u + 2$*, therefore we can write* $\mathbf{D}$ *in one unknown as* $(u, 2u + 2)$ *and solve directly from the cross-ratio.]*

□

Question 4 Consider the conic sections

$$C = \left\{(u, v) \in \mathbb{A}^2 \;\middle|\; \frac{u^2}{4} + \frac{v^2}{9} = 1\right\}, \quad D = \left\{(u, v) \in \mathbb{A}^2 \;\middle|\; u^2 - 4v^2 = 1\right\}.$$

Calculate a projective transformation T that transforms C to D on the third affine chart.

Solution. The projective transformation that maps the unit circle to D (as given in the proof of Theorem 5.22) is the transformation $T([x : y : z]) = [z : \frac{y}{2} : x]$ given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The projective transformation that maps the unit circle to C is the transformation $S([x : y : z]) = [2x : 3y : z]$. Therefore the inverse transformation S^{-1} maps C to the unit circle, and its matrix is given by the inverse matrix to S:

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transformation that maps C to D is $T \circ S^{-1}$: it maps C to the unit circle, then the unit circle to D. Multiplying matrices, we see this is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/6 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$
$$\Rightarrow T \circ S^{-1}([x : y : z]) = \left[z : \frac{y}{6} : \frac{x}{2}\right] = [6z : y : 3x]$$

□

2 Extra Exercises

Question 5 Let $T\colon \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation that sends

$$T([0:1:2]) = \mathbf{A}, \quad T([0:2:1]) = \mathbf{B}, \quad T([0:0:1]) = \mathbf{C}.$$

For each set of points $\mathbf{A}, \mathbf{B}, \mathbf{C}$, find the point $\mathbf{X} \in \mathbb{P}^2$ such that $T(\mathbf{X}) = \mathbf{D}$.

(1) $\mathbf{A} = [2:3:0]$, $\mathbf{B} = [1:3:0]$, $\mathbf{C} = [1:1:0]$, $\mathbf{D} = [1:2:0]$

(2) $\mathbf{A} = [2:1:4]$, $\mathbf{B} = [2:1:1]$, $\mathbf{C} = [0:0:1]$, $\mathbf{D} = [2:1:0]$

Solution. Note none of the listed points are on $\mathbb{A}^2_1$, therefore they lie on the copy of $\mathbb{P}^1$ disjoint from this affine chart:

$$\mathbb{P}^1 = \{[0:y:z] \mid y, z \text{ not both zero}\}$$

We can consider the (1-dimensional) affine chart $\mathbb{A}^1 = \{[0:v:1] \mid v \in \mathbb{R}\} \subset \mathbb{P}^1$ with affine coordinate v. Then the cross ratio of the listed points and $\mathbf{X} = [0:v:1]$ is

$$\frac{(1/2 - 0)(2 - v)}{(1/2 - v)(2 - 0)} = \frac{2 - v}{2 - 4v}$$

As projective transformations preserve the cross ratio, we must pick v such that this cross ratio equals $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$.

(1)

$$(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{(3/2 - 1)(3 - 2)}{(3/2 - 2)(3 - 1)} = -1/2$$
$$\frac{2 - v}{2 - 4v} = \frac{-1}{2} \quad \Rightarrow \quad v = 1$$

Therefore $\mathbf{X} = [0:1:1]$.

(2)

$$(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{(4 - \infty)(1 - 0)}{(4 - 0)(1 - \infty)} = \frac{1}{4}$$
$$\frac{2 - v}{2 - 4v} = \frac{1}{4} \quad \Rightarrow \quad v = \infty$$

Therefore $\mathbf{X} = [0:1:0]$.

□

Question 6 Find a condition on the ordering of four collinear points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ such that $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) < 0$.

Solution. As the points are collinear, they lie on a copy of $\mathbb{P}^1$. Therefore we can describe the points in affine coordinates $u \in \mathbb{R} \cup \{\infty\}$.

$\frac{\mathbf{A}-\mathbf{C}}{\mathbf{A}-\mathbf{D}}$ is positive if $\mathbf{A} > \mathbf{C}, \mathbf{D}$ or $\mathbf{A} < \mathbf{C}, \mathbf{D}$. Conversely it is negative if $\mathbf{C} < \mathbf{A} < \mathbf{D}$ or $\mathbf{D} < \mathbf{A} < \mathbf{C}$, i.e. $\mathbf{A}$ is between $\mathbf{C}$ and $\mathbf{D}$. We get a similar condition for $\frac{\mathbf{B}-\mathbf{D}}{\mathbf{B}-\mathbf{C}}$.

As

$$(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{\mathbf{A} - \mathbf{C}}{\mathbf{A} - \mathbf{D}} \cdot \frac{\mathbf{B} - \mathbf{D}}{\mathbf{B} - \mathbf{C}},$$

we get that $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) < 0$ if exactly one of $\frac{\mathbf{A}-\mathbf{C}}{\mathbf{A}-\mathbf{D}}$ and $\frac{\mathbf{B}-\mathbf{D}}{\mathbf{B}-\mathbf{C}}$ is negative. This happens only when exactly one of $\mathbf{A}, \mathbf{B}$ is between $\mathbf{C}$ and $\mathbf{D}$. □