Exercises 10

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following projective transformations, write down the induced transformation on each affine chart. State whether the induced transformation on each chart is affine or rational.

(1) $T\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad T([x:y]) = [5x : 2x + 3y]$

(2) $T\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad T([x:y:z]) = [x + y : 2y : y + z]$

Solution. (1) On the first affine chart, T acts as

$$T([1:v]) = [5 : 3v + 2] = \left[1 : \frac{3v+2}{5}\right].$$

In affine coordinates v, this is the affine transformation $v \mapsto \frac{3}{5}v + \frac{2}{5}$.

On the second affine chart, T acts as

$$T([u:1]) = [5u : 2u + 3] = \left[\frac{5u}{2u+3} : 1\right].$$

In affine coordinates u, this is the rational transformation $u \mapsto \frac{5u}{2u+3}$ as the transformation is undefined for $u = \frac{-2}{3}$.

(2) On the first affine chart, T acts as

$$T([1:v:w]) = [1 + v : 2v : v + w] = \left[1 : \frac{2v}{v+1} : \frac{v+w}{v+1}\right].$$

In affine coordinates (v, w), this is the rational transformation $(v, w) \mapsto (\frac{2v}{v+1}, \frac{v+w}{v+1})$ as the transformation is undefined for $v = -1$.

On the second affine chart, T acts as

$$T([u:1:w]) = [u + 1 : 2 : 1 + w] = \left[\frac{u+1}{2} : 1 : \frac{w+1}{2}\right].$$

In affine coordinates (u, w), this is the affine transformation $(u, w) \mapsto (\frac{u+1}{2}, \frac{w+1}{2})$.

On the third affine chart, T acts as

$$T([u:v:1]) = [u+v:2v:v+1] = \left[\frac{u+v}{v+1} : \frac{2v}{v+1} : 1\right].$$

In affine coordinates (u,v), this is the rational transformation $(u,v) \mapsto (\frac{u+v}{v+1}, \frac{2v}{v+1})$ as the transformation is undefined for $v=-1$.

□

Question 2 For each of the following affine/rational transformations, find a projective transformation that induces the transformation on some affine chart.

(1) $f(u,v) = (3v, 2u+1)$

(2) $f(u,v) = \left(\frac{3v}{2u+v+1}, \frac{u+v}{2u+v+1}\right)$

Solution. For both, let us assume these are the induced transformations on the third affine chart. Therefore we find the image of $[x:y:z]$ where $u = \frac{x}{z}, v = \frac{y}{z}$.

(1)

$$\begin{aligned} T([x:y:z]) &= T([u:v:1]) \\ &= [3v : 2u+1 : 1] \\ &= [3y : 2x+z : z] \end{aligned}$$

(2)

$$\begin{aligned} T([x:y:z]) &= T([u:v:1]) \\ &= \left[\frac{3v}{2u+v+1} : \frac{u+v}{2u+v+1} : 1\right] \\ &= \left[\frac{3y}{2x+y+z} : \frac{x+y}{2x+y+z} : 1\right] \\ &= [3y : x+y : 2x+y+z] \end{aligned}$$

□

2 Extra Exercises

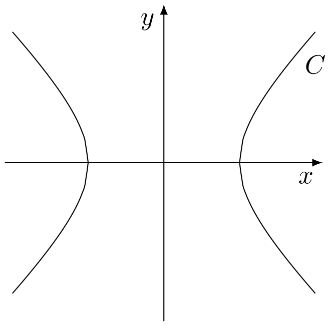
Question 3 Let (x, y) be a Cartesian coordinate system for $\mathbb{A}^2$. Let C be the curve with implicit equation $x^2 - y^2 = 1$. Let D be the curve with equation $2xy = 1$.

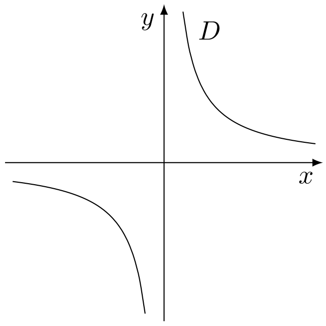
(1) Sketch the two curves C and D.

(2) Find (with proof) an affine transformation that maps C to D.

(3) Deduce that D is a hyperbola.

Solution. (1)





(2) A rotation about the origin by $\frac{\pi}{4}$ maps C to D:

Proof: The affine transformation has the matrix

$$[f] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $(x, y) \mapsto \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$. Thus

$$\begin{aligned} & (x, y) \in C \\ \iff \quad & x^2 - y^2 = 1 \\ \iff \quad & (x + y)(x - y) = 1 \\ \iff \quad & 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 1 \\ \iff \quad & f((x, y)) \in D \end{aligned}$$

□

(3) Since C is a hyperbola and D is the affine image of a C, we must have that D is a hyperbola.

□

Question 4 Let (x, y) be a Cartesian coordinate system for $\mathbb{A}^2$ and let C be the curve with implicit equation $y = ax^2 + bx + c$ for $a, b, c \in \mathbb{R}$ and $a \neq 0$. By considering the affine transformation f with matrix

$$\begin{bmatrix} 0 & \frac{1}{a} & \frac{b^2-4ac}{4a^2} \\ 1 & 0 & \frac{b}{2a} \\ 0 & 0 & 1 \end{bmatrix}$$

prove that C is a parabola.

Solution. Under the affine transformation we have that

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \overset{f}{\longmapsto} \begin{bmatrix} 0 & \frac{1}{a} & \frac{b^2-4ac}{4a^2} \\ 1 & 0 & \frac{b}{2a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{y}{a} + \frac{b^2-4ac}{4a^2} \\ x + \frac{b}{2a} \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{bmatrix}$$

Then

$$\begin{aligned}
& (x,y) \in C \\
\iff \quad & y = ax^2 + bx + c \\
\iff \quad & \tfrac{y}{a} = x^2 + \tfrac{b}{a}x + \tfrac{c}{a} \\
\iff \quad & \tfrac{y}{a} = \left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a^2} + \tfrac{c}{a} \\
\iff \quad & \left(x + \tfrac{b}{2a}\right)^2 = \tfrac{y}{a} + \tfrac{b^2-4ac}{4a^2} \\
\iff \quad & \tilde{y}^2 = \tilde{x}.
\end{aligned}$$

Since the curve $f(C)$ is a parabola we know that C must also be a parabola. □

Question 5 In the course we have shown how to write $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^0$, where $\mathbb{A}^0$ is a point at infinity.

Show that one can write $\mathbb{P}^n$ as

$$\mathbb{P}^n = \bigcup_{i=0}^{n} \mathbb{A}^i.$$

Solution. We saw that we can break $\mathbb{P}^n$ into a copy of $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ by considering points where the final coordinate is non-zero:

$$\begin{aligned}
\mathbb{P}^n &= \{[x_1 : \cdots : x_n : x_{n+1}] \mid x_{n+1} \neq 0\} \cup \{[x_1 : \cdots : x_n : x_{n+1}] \mid x_{n+1} = 0\} \\
&= \{[u_1 : \cdots : u_n : 1] \mid u_i \in \mathbb{R}\} \cup \{[x_1 : \cdots : x_n : 0] \mid x_i \in \mathbb{R}\,(\text{not all zero})\} \\
&= \mathbb{A}^n \cup \mathbb{P}^{n-1}.
\end{aligned}$$

We can repeat this decomposition on $\mathbb{P}^{n-1}$ to get $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2}$. Repeating until we have decomposed all $\mathbb{P}^i$ until $\mathbb{P}^0 = \mathbb{A}^0$, we get

$$\begin{aligned}
\mathbb{P}^n &= \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \mathbb{A}^1 \cup \mathbb{A}^0 \\
&= \{[u_1 : \cdots : u_n : 1]\} \cup \{[u_1 : \cdots : u_{n-1} : 1 : 0]\} \cup \cdots \\
&\quad \cdots \cup \{[u_1 : 1 : 0 : \cdots : 0]\} \cup \{[1 : 0 : \cdots : 0]\}
\end{aligned}$$

(To be rigorous with this proof, one should use induction but the argument is exactly the same). □

Question 6 Let $T\colon \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation such that the induced transformation on each affine chart is an affine transformation. Show that T is of the form

$$T([x_1 : \cdots : x_{n+1}]) = [\lambda_1 x_1 : \cdots : \lambda_{n+1} x_{n+1}], \quad \lambda_i \in \mathbb{R} \setminus \{0\}.$$

Solution. Let T be a projective transformation that induces an affine transformation on each affine chart. Let $M = (M_{i,j})$ be the $(n+1) \times (n+1)$ matrix that induces this transformation.

Consider the point $\mathbf{P}_j = [0 : \cdots : 1 : \cdots : 0]$ with zeroes everywhere except the j-th entry. Furthermore, we have

$$T(\mathbf{P}_j) = [M_{1,j} : \cdots : M_{n+1,j}].$$

As T induces an affine map on $\mathbb{A}^n_j$, it must map affine points to affine points. Therefore as $\mathbf{P}_j \in \mathbb{A}^n_j$, we have $T(\mathbf{P}_j) \in \mathbb{A}^n_j$ and so $M_{j,j}$ is nonzero for all j.

Suppose $M_{i,j} \neq 0$ for $i \neq j$. Then we consider the point $\mathbf{Q}$ with $\frac{-M_{j,j}}{M_{i,j}}$ in the ith entry, 1 in the jth entry and zero everywhere else. Then in the j-th entry of $T(\mathbf{Q})$ we have

$$T(\mathbf{Q})_j = \frac{-M_{j,j}}{M_{i,j}} M_{i,j} + M_{(j,j)} = 0\,,$$

implying that $T(\mathbf{Q})$ is not on the jth affine chart, and that T induces a rational transformation, a contradiction.

Therefore M is a diagonal matrix - if the elements on its diagonal are λ_j, we get exactly the projective transformation described.

(This is almost exactly the proof of Proposition 5.13). □