Exercises 9

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$ and let C be the curve with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(1) Calculate the focus points for the curve C.

(2) Specify focal polar coordinates for the curve C and write down an equation for the curve in these coordinates.

Question 2 Fix a Cartesian coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ for $\mathbb{A}^n$. Let $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ and $g(\mathbf{x}) = G(\mathbf{x}) + \mathbf{Q}$ be two affine transformations. [So F and G are invertible linear maps.]

(1) Prove that the composition $h := f \circ g$ is also an affine transformation $h(\mathbf{x}) = H(\mathbf{x}) + \mathbf{R}$. *[Hint: $h(\mathbf{x}) = f(g(\mathbf{x}))$.]*

(2) Let h denote the identity transformation $h(\mathbf{x}) = \mathbf{x}$. Suppose $h = f \circ g$ is the identity. What are G and $\mathbf{Q}$ in terms of F and $\mathbf{P}$?

[In particular, the above shows that the set of affine transformations is a *group*, denoted AGL_n.]

Question 3 Let f be the affine map $f : \mathbb{A}^2 \to \mathbb{A}^2$ that is a reflection in the line $x = 1$. Suppose $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ for a linear invertible map F and vector $\mathbf{P} \in \mathbb{A}^2$.

(1) Determine the vector $\mathbf{P}$.

(2) Work out where f sends the points $(1, 0)$ and $(0, 1)$. Hence, write down the matrix of $F\colon \mathbb{E}^2 \to \mathbb{E}^2$ with respect to the standard basis on $\mathbb{A}^2$.

Question 4 Consider the affine transformation $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ on $\mathbb{A}^2$ that is made up of a rotation F by $\frac{\pi}{3}$ about the origin, followed by a translation by $\mathbf{P} = (3, 2)$.

(1) Write down the matrix of the linear operator F with respect to the standard basis.

(2) Find a point fixed by f.

(3) Describe the affine transformation f in a coordinate system where the fixed point is taken to be the origin.

Question 5 Let $f(\mathbf{x}) = F(\mathbf{x}) + \mathbf{P}$ be the affine transformation with

$$F = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}$$

(1) Find a line in $\mathbb{A}^2$ that is fixed by f.

(2) Show that the induced linear operator F of f is orthogonal.

(3) Deduce that f is a reflection in the fixed line.