Exercises 8

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following curves C and 1-forms ω, calculate $\int_C \omega$ up to sign.

(1) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid y = x^2,\, 0 < x < 1 \right\}, \quad \omega = y\mathbf{d}x - x\mathrm{d}y$

(2) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid y^3 = x,\, -1 < x < 1 \right\}, \quad \omega = xy\mathbf{d}x + xy\mathbf{d}y$

(3) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid x^2 + y^2 = 13 \right\}, \quad \omega = (2xy + y^2)\mathbf{d}x + (x^2 + 2xy)\mathbf{d}y$

(4) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \mid x^2 - y^2 = 1,\, 1 < x < 2,\, y > 0 \right\}, \quad \omega = e^x\mathbf{d}x + \mathbf{d}y$

Question 2 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be an ellipse with foci $\mathbf{F}_1 = (-1, 0)$ and $\mathbf{F}_2 = (1, 0)$. The point $\mathbf{P} = (0, 2)$ is on the ellipse C.

(1) Find the implicit equation for the ellipse in Cartesian coordinates.

(2) What is the eccentricity, e, of the ellipse.

Question 3 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be a hyperbola with focus points $\mathbf{F}_1 = (-2, 0)$ and $\mathbf{F}_2 = (2, 0)$ and assume the point $\mathbf{P} = (2, 3)$ is on the curve C.

(1) Find the two points $(s, 0)$ and $(t, 0)$ where C intersects the x-axis.

(2) Write down the implicit equation of the curve.

2 Extra Exercises

Question 4 Consider the curve

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x = 2y^2 - 1, 0 < y < 1 \right\}.$$

(1) Compute two parametrisations of C with the opposite orientation.

(2) Calculate the integral

$$\int_C \sin y \mathbf{d}x$$

for each parametrisation.

Question 5 Calculate the integral

$$\int_C x\mathbf{d}y$$

for each of the following curves:

(1) the circle $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 + y^2 = 12y \right\}$

(2) the ellipse $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 + \frac{y^2}{9} = 1 \right\}$.

Question 6 Let C be the circle with radius a centred at the origin. Recall that we can parametrise C via

$$\begin{aligned} \boldsymbol{\gamma}\colon [0, 2\pi) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}. \end{aligned}$$

(1) Write $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ in polar coordinates (r, θ).

(2) Compute the integral

$$\int_C r^2 \mathbf{d}\theta.$$

(3) How does the integral in part (b) relate to the following integral?

$$\int_C -y\mathbf{d}x + x\mathbf{d}y$$

Question 7 Fix a Cartesian coordinate system (x, y) for $\mathbb{A}^2$. Let C be an ellipse with focus points $\mathbf{F}_1 = (-5, 0)$ and $\mathbf{F}_2 = (16, 0)$. The point $\mathbf{P} = (0, 12)$ is on the ellipse.

(1) Find the two points of intersection between C and the line $x = 0$ (that is, the y-axis).

(2) Find the two points of intersection between C and the line $y = 0$ (that is, the x-axis).

(3) What is the eccentricity of the ellipse?

Question 8 Fix Cartesian coordinates (x, y) and let C_1, C_2 and C_3 be conic sections with the implicit equations:

$$\begin{aligned} C_1 &\colon 4x^2 + 4x + y^2 = 0; \\ C_2 &\colon 4x^2 + 4x - y^2 = 0; \\ C_3 &\colon 4x^2 + 4x + y = 0. \end{aligned}$$

Show that C_1 is an ellipse, C_2 is a hyperbola and C_3 is a parabola.