Exercises 7

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 For each of the following parametrisations $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$

(1) show $\boldsymbol{\gamma}_1$ is a reparametrisation of $\boldsymbol{\gamma}_2$,

(2) state whether they have the same orientation.

$$\begin{aligned} \boldsymbol{\gamma}_1 \colon (0,1) &\to \mathbb{A}^2 & \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{4}\right) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix} & t &\mapsto \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} \end{aligned} \tag{1}$$

$$\begin{aligned} \boldsymbol{\gamma}_1 \colon (0,1) &\to \mathbb{A}^2 & \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{2}\right) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} t \\ 2t^2-1 \end{pmatrix} & t &\mapsto \begin{pmatrix} \cos t \\ \cos 2t \end{pmatrix} \end{aligned} \tag{2}$$

$$\begin{aligned} \boldsymbol{\gamma}_1 \colon (0,1) &\to \mathbb{A}^2 & \boldsymbol{\gamma}_2 \colon \left(0, \frac{\pi}{2}\right) &\to \mathbb{A}^2 \\ t &\mapsto \begin{pmatrix} 1-t \\ 1-2t \end{pmatrix} & t &\mapsto \begin{pmatrix} \sin^2 t \\ -\cos 2t \end{pmatrix} \end{aligned} \tag{3}$$

Question 2 Consider the curve C

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \;\middle|\; y^2 = x^3 + 1,\ -1 \leq x \leq 1,\ y > 0 \right\},$$

and the point on the curve $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(1) By calculating $\frac{\mathrm{d}y}{\mathrm{d}x}$, make a prediction what the tangent space $T_{\mathbf{P}}(C)$ to the curve at $\mathbf{P}$ will be.

(2) For each of the following vectors, find a parametrisation $\boldsymbol{\gamma}$ of C such that it is the velocity vector $\boldsymbol{\gamma}'$ at $\mathbf{P}$:

$$\text{(i)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 Extra Exercises

Question 3 Consider a 0-form $f(x, y)$ on $\mathbb{A}^2$. In terms of partial derivatives of f:

(1) find a vector field $\mathbf{W}$ such that $D_{\mathbf{W}} f$ is zero for all points in $\mathbb{A}^2$,

(2) find a parametrisation $\boldsymbol{\gamma}$ of a curve C such that $D_{\boldsymbol{\gamma}'} f$ is zero for all points in C.

Question 4 For each of the following implicit curves, find two parametrisations whose orientations are opposite.

(1) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \ \middle| \ \frac{x^2}{3} + \frac{y^2}{5} = 1 \right\}$

(2) $C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \ \middle| \ x^2 - y^2 = 1 \right\}$,