Exercises 6

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Plot each of the following vector fields in $\mathbb{A}^2$. Determine whether they are conservative or not.

(1) $\mathbf{W} = -x\mathbf{e}_x - y\mathbf{e}_y$

(2) $\mathbf{W} = y\mathbf{e}_x + x\mathbf{e}_y$

(3) $\mathbf{W} = (x+y)\mathbf{e}_x + (x+y)\mathbf{e}_y$

Question 2 For each of the following 1-forms ω, find all points $\mathbf{P} \in \mathbb{A}^2$ such that $\omega_{\mathbf{P}} := \omega(\mathbf{P}, -)$ is the linear functional

$$\begin{aligned} \omega_{\mathbf{P}} \colon T_{\mathbf{P}}\left(\mathbb{A}^2\right) &\to \mathbb{R} \\ \begin{bmatrix} v_x \\ v_y \end{bmatrix} &\mapsto v_x + v_y. \end{aligned}$$

(1) $\omega = x\mathbf{d}x + y\mathbf{d}y$

(2) $\omega = 3x\mathbf{d}x + y^2\mathbf{d}y$

(3) $\omega = \sin x\mathbf{d}x + \cos y\mathbf{d}y$

Question 3 Which of the following 1-forms are exact? If they are exact, find a 0-form f such that $\omega = \mathbf{d}f$. If they are not exact, prove no such 0-form exists.

(1) $\omega = \sin y\mathbf{d}x + x\cos y\mathbf{d}y$

(2) $\omega = 2x\mathbf{d}x + 3y^2\mathbf{d}y$

(3) $\omega = x^3\mathbf{d}y$

Question 4 Consider the 0-form $f = x^3 - y^3$ and the vector field $\mathbf{W} = -y\mathbf{e}_x + x\mathbf{e}_y$.

(1) Calculate the differential of f.

(2) Compute $\mathbf{d}f(-, \mathbf{W})$.

(3) Using your previous answer, find all points in $\mathbb{A}^2$ where the directional derivative $D_{\mathbf{W}}f$ of f along $\mathbf{W}$ is zero.

2 Extra Exercises

Question 5 We shall show that 1-form computations do not depend on the coordinate system by considering a computation in polar coordinates (r, θ).

(1) Consider the vector field $\mathbf{W} = -y\mathbf{e}_x + x\mathbf{e}_y$. Show that we can rewrite $\mathbf{W}$ in polar coordinates as

$$\mathbf{W} = \mathbf{e}_\theta.$$

(2) Polar coordinates have the elementary 1-forms $\mathbf{d}r, \mathbf{d}\theta$ where for a vector $\mathbf{v} = [v_r, v_\theta]^\mathsf{T}$ we have

$$\mathbf{d}r(\mathbf{v}) = v_r, \quad \mathbf{d}\theta(\mathbf{v}) = v_\theta.$$

Furthermore, they are related to the elementary forms $\mathbf{d}x, \mathbf{d}y$ by

$$\mathbf{d}x = \frac{\partial x}{\partial r}\mathbf{d}r + \frac{\partial x}{\partial \theta}\mathbf{d}\theta, \quad \mathbf{d}y = \frac{\partial y}{\partial r}\mathbf{d}r + \frac{\partial y}{\partial \theta}\mathbf{d}\theta.$$

Show that we can rewrite the 1-form $\omega = x\mathbf{d}x + y\mathbf{d}y$ in polar coordinates as

$$\omega = r\mathbf{d}r.$$

(3) Calculate $\omega(-, \mathbf{W})$ in polar coordinates. How does your solution relate to the same calculation in Cartesian coordinates? (Example 2.29 in the notes)