Exercises 5

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

For all questions, we work in Cartesian coordinates $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{A}^2$, where each tangent space has basis $(\mathbf{e}_x, \mathbf{e}_y) \subset \mathbb{E}^2$.

Question 1 Calculate the area of the parallelogram $\square(\mathbf{a}, \mathbf{b})$, formed by the following vectors, where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis.

(1) $\mathbf{a} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$;

(2) $\mathbf{a} = 5\mathbf{e}_1 + 8\mathbf{e}_2 + 4\mathbf{e}_3$ and $\mathbf{b} = 10\mathbf{e}_1 + 16\mathbf{e}_2 + 8\mathbf{e}_3$.

Question 2 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$ and let P be a linear operator with the matrix: $[P]_\mathcal{B} = \left[\begin{smallmatrix} 5 & -1 \\ 2 & 2 \end{smallmatrix}\right]$. (See Exercise Sheet 2 Question 8.)

(1) Calculate the area of the parallelogram $\square(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2)$.

(2) Recall the $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$ is an eigenvector of P with eigenvalue 4; and $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2$ is an eigenvector of P with eigenvalue 3. Calculate the area of the image parallelogram $\square(P(\mathbf{a}), P(\mathbf{b}))$ without using the determinant.

(3) Compare your answers to parts (1) and (2) with the $\det P$.

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis for $\mathbb{E}^3$. Let P be the linear operator with matrix $[P]_\mathcal{B} = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

(1) Calculate the volume of the parallelepiped formed by the vectors $\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_3$; $\mathbf{b} = 4\mathbf{e}_2 + 3\mathbf{e}_3$; and $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2$.

(2) Calculate $P(\mathbf{a})$, $P(\mathbf{b})$ and $P(\mathbf{c})$.

(3) What is the volume of the parallelepiped $\square(P(\mathbf{a}), P(\mathbf{b}), P(\mathbf{c}))$? (without using the determinant)

(4) Compare the volumes from parts (1) and (3) with the determinant of P.

2 Extra Exercises

Question 4 Let $\mathcal{B}$, P, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be as in Question 3.

(1) What is the area of the parallelograms:

(a) $\varnothing(\mathbf{a}, \mathbf{b})$; (b) $\varnothing(\mathbf{a}, \mathbf{c})$?

(2) Compare these to the areas of the image parallelograms:

(a) $\varnothing(P(\mathbf{a}), P(\mathbf{b}))$; (b) $\varnothing(P(\mathbf{a}), P(\mathbf{c}))$?

(3) Can you say anything about the determinant of a linear operator acting on a 3-dimensional space and the scaling of parallelograms?

(4) Can you find a linear operator that fixes the volume of parallelepipeds but scales the area of some parallelogram by λ?

Question 5 Consider the set of points

$$L = \left\{ \begin{pmatrix} x \\ mx + c \end{pmatrix} \in \mathbb{A}^2 \ \mid \ x \in \mathbb{R} \right\}, \ m, c \in \mathbb{R}.$$

Show that L is an affine space.

[Hint: As L forms a line, the associated vector space should be one-dimensional.]

Question 6 Fix Cartesian coordinates (x, y) on $\mathbb{A}^2$ with respect to an origin $\mathbf{O}$ and orthonormal basis $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$.

(1) One way to define polar coordinates (r, θ) on $\mathbb{A}^2$ is via the map

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{A}^2 \\ (r, \theta) &\mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \end{aligned}$$

Show this map is a coordinate system. *[Hint: It may be helpful to recall the following trigonometric identities:]*

$$\sin(\arctan(z)) = \frac{z}{\sqrt{1+z^2}}, \quad \cos(\arctan(z)) = \frac{1}{\sqrt{1+z^2}}$$

(2) Polar coordinates give rise to a different basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ for the tangent spaces $T_{\mathbf{P}}\left(\mathbb{A}^2\right)$ via the equations:

$$\mathbf{e}_r = \frac{\partial x}{\partial r}\mathbf{e}_x + \frac{\partial y}{\partial r}\mathbf{e}_y, \quad \mathbf{e}_\theta = \frac{\partial x}{\partial \theta}\mathbf{e}_x + \frac{\partial y}{\partial \theta}\mathbf{e}_y.$$

Write down the basis vectors in terms of $\mathbf{e}_x, \mathbf{e}_y$. Is this basis orthonormal?

(3) Describe how the basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ for $T_{\mathbf{P}}\left(\mathbb{A}^2\right)$ varies as we vary $\mathbf{P}$. What is the connection between (r, θ) and $(\mathbf{e}_r, \mathbf{e}_\theta)$?