Exercises 4

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Define a linear operator P on $\mathbb{E}^3$ by

$$\begin{aligned} P(\mathbf{e}) &= \mathbf{e} \\ P(\mathbf{f}) &= \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g} \\ P(\mathbf{g}) &= -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g} \end{aligned}$$

(1) Write down the matrix $[P]_{\mathcal{B}}$.

(2) Show P is an orthogonal operator that preserves orientation.

(3) Find the axis and angle of rotation.

Question 2 Let $\mathcal{B} = (\mathbf{e}, \mathbf{f}, \mathbf{g})$ be an orthonormal basis in $\mathbb{E}^3$. Consider the linear operator P_1 in $\mathbb{E}^3$ defined by

$$P_1(\mathbf{e}) = \mathbf{f} \qquad P_1(\mathbf{f}) = \mathbf{e} \qquad P_1(\mathbf{g}) = \mathbf{g}.$$

Also consider the linear operator P_2 that is the reflection operator in the plane spanned by $\mathbf{e}$ and $\mathbf{f}$.

(1) State whether P_1 and P_2 preserves orientation.

(2) Define the operator $P = P_1 \circ P_2$ that applies the operator P_2 followed by the operator P_1. Does P preserve orientation?

(3) Show that P is a rotation operator. Find its axis and angle of rotation.

2 Extra Exercises

Question 3 Let $\mathbf{n} \in \mathbb{E}^3$ be a unit vector and consider the following operators on $\mathbb{E}^3$

$$P_1(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n}, \quad P_2(x) = 2\langle \mathbf{n}, \mathbf{x}\rangle \mathbf{n} - \mathbf{x}.$$

(1) Show both operators are orthogonal.

(2) Show the first operator is a reflection, along with the plane it is a reflection through.

(3) Show the second operator is a rotation. Find its axis and angle of rotation.

[(Almost) all of this question can be done without writing a matrix! You may find it helpful to consider the plane $H_{\mathbf{n}}$ whose normal vector is $\mathbf{n}$:[1]]

$$H_{\mathbf{n}} = \left\{\mathbf{x} \in \mathbb{E}^3 \;\middle|\; \langle \mathbf{n}, \mathbf{x}\rangle = 0\right\}$$

Question 4 † In this question we will prove the determinant formula satisfies the axioms of a vector product. Fix an orthogonal basis $\mathcal{B}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $\mathbb{E}^3$, which we use to define the orientation of $\mathbb{E}^3$. For any two vectors $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ define the function $- \times -\colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ by the determinant formula:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

(VP-AC) Show, or explain why, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

((VP-⊥)) (1) Show that $\langle \mathbf{v} \times \mathbf{w}, \mathbf{v}\rangle = 0$ for all vectors $\mathbf{v}$ and $\mathbf{w}$.

(2) Using (VP-AC) deduce that $\langle \mathbf{v} \times \mathbf{w}, \mathbf{w}\rangle = 0$.

(VP-Lin) (1) Show the following identity holds:

$$\det \begin{bmatrix} \lambda a + \mu a' & \lambda b + \mu b' \\ c & d \end{bmatrix} = \lambda \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mu \det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}.$$

(2) Prove that $(\lambda \mathbf{v} + \mu \mathbf{w}) \times \mathbf{x} = \lambda(\mathbf{v} \times \mathbf{x}) + \mu(\mathbf{w} \times \mathbf{x})$ for all vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

(VP-Len) Recall the identity of Lemma 1.88: $\|\mathbf{v} \times \mathbf{w}\|^2 + \langle \mathbf{v}, \mathbf{w}\rangle^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$.

Using this formula, or otherwise, prove $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\|$ for all perpendicular vectors $\mathbf{v}$ and $\mathbf{w}$.

(VP-O) Let $v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$, be linearly independent vectors.

(1) Let $\mathcal{C} = (\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w})$, which you may assume is a basis for $\mathbb{E}^3$. Write down the transition matrix T, from basis $\mathcal{B}$ to basis $\mathcal{C}$.

(2) Calculate a formula for the determinant of T and deduce that $\mathcal{B}$ and $\mathcal{C}$ have the same orientation.
It may be helpful to use the 3^{rd} *column to calculate the determinant:*

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} - f \det \begin{bmatrix} a & b \\ g & h \end{bmatrix} + i \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

Finally: Deduce that $- \times -$ is a vector product as defined in Definition 1.82.

Question 5 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis in $\mathbb{E}^3$. Find a unit vector $\mathbf{n}$, such that the following conditions hold:

(1) $\mathbf{n}$ is orthogonal to $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$;

(2) $\mathbf{n}$ is orthogonal to $\mathbf{b} = \mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3$;

[1] In general, any plane in $\mathbb{E}^3$ (or $(n-1)$-dimensional *hyperplane* in $\mathbb{E}^n$) can be written as the set of vectors orthogonal to a normal vector $\mathbf{n}$.

(3) The basis $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ has an orientation opposite to $\mathcal{B}$.

Express $\mathbf{n}$ using the basis $\mathcal{B}$.

Question 6 Elisa and Faraz calculate the vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{E}^3$. Elisa uses the orthonormal basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and Faraz use the orthonormal basis $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. The vectors expressed in these bases are:

$$[\mathbf{a}]_{\mathcal{E}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{E}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad [\mathbf{a}]_{\mathcal{F}} = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} \qquad [\mathbf{b}]_{\mathcal{F}} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

They both use the *determinant formula* for calculations:

$$\mathbf{a} \times \mathbf{b} \stackrel{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}}_{\text{Elisa's calculations}} \stackrel{?}{=} \underbrace{\det \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{bmatrix}}_{\text{Faraz's calculations}}.$$

(1) In what situations do their calculations give the same vector, and when do they give different answers?

(2) In each case what can you say about the linear operator P, that maps each vector $\mathbf{e}_i \mapsto \mathbf{f}_i$?