Exercises 3

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

Question 1 (1) Group the following bases into equivalent classes with respect to orientation. That is, all bases in the same class must share an orientation and any pair in different classes must have a different orientation.

$$\mathcal{B}_1 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \qquad \mathcal{B}_2 = (\mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_x) \qquad \mathcal{B}_3 = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)$$
$$\mathcal{B}_4 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_y - \mathbf{e}_x, \mathbf{e}_z) \qquad \mathcal{B}_5 = (\mathbf{e}_y + \mathbf{e}_z, \mathbf{e}_z - \mathbf{e}_y, \mathbf{e}_x) \qquad \mathcal{B}_6 = (\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x - \mathbf{e}_y, \mathbf{e}_z)$$

(2) Using your answer to part (1), or otherwise, determine which of the following linear operators preserve the orientation and which change the orientation.

$$\begin{array}{llll}
P_1 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_1(\mathbf{e}_x) = \mathbf{e}_y, & P_1(\mathbf{e}_y) = \mathbf{e}_z, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\
P_2 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_2(\mathbf{e}_x) = \mathbf{e}_y, & P_2(\mathbf{e}_y) = \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\
P_3 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_3(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_3(\mathbf{e}_y) = \mathbf{e}_y - \mathbf{e}_x, & P_3(\mathbf{e}_z) = \mathbf{e}_z \\
P_4 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_4(\mathbf{e}_x) = \mathbf{e}_y + \mathbf{e}_z, & P_4(\mathbf{e}_y) = \mathbf{e}_z - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_x \\
P_5 \colon \mathbb{E}^3 \to \mathbb{E}^3 & P_5(\mathbf{e}_x) = \mathbf{e}_x + \mathbf{e}_y, & P_5(\mathbf{e}_y) = \mathbf{e}_x - \mathbf{e}_y, & P_3(\mathbf{e}_z) = \mathbf{e}_z
\end{array}$$

Question 2 Consider the matrices

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$$

For each matrix:

(1) find the eigenvalues λ by finding the roots of the polynomial $\det(M - \lambda I)$,

(2) calculate the corresponding eigenvectors.

Question 3 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Recall that the reflection operator R in the line spanned by $\mathbf{e}_1$ has matrix

$$[R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\mathbf{x}_\varphi = P_\varphi(\mathbf{e}_1)$ be the vector obtained by rotating $\mathbf{e}_1$ by φ. Let R_φ be the linear operator that is the reflection operator in the line spanned by $\mathbf{x}_\varphi$.

(1) Write R_φ as a composition of rotation operators and the reflection operator R.

(2) Show that $R_\varphi = Q_{2\varphi}$.

2 Extra Exercises

Question 4 Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{E}^2$, recall that P_φ is the operator that rotates by φ. Show the following identities for matrices of rotation operators in $\mathbb{E}^2$:

(1) $[P_{-\varphi}]_\mathcal{B} = ([P_\varphi]_\mathcal{B})^\mathsf{T} = ([P_\varphi]_\mathcal{B})^{-1}$

(2) $[P_\varphi]_\mathcal{B}[P_\theta]_\mathcal{B} = [P_{\varphi+\theta}]_\mathcal{B}$

Question 5 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{E}^2$. Consider the linear operators P_1, P_2 on $\mathbb{E}^2$ defined by

$$P_1(\mathbf{e}_1) = \mathbf{e}_1 \qquad P_2(\mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2$$
$$P_1(\mathbf{e}_2) = \mathbf{e}_1 + \mathbf{e}_2 \qquad P_2(\mathbf{e}_2) = \mathbf{e}_2$$

(1) Write down the matrices for P_1, P_2. Which of them are orthogonal operators?

(2) Find all linear operators of the form $P = aP_1 + bP_2$ that are orthogonal (where $a, b \in \mathbb{R}$).

(3) For each orthogonal P, write it as either a rotation operator P_φ or a reflection operator Q_φ.