Exercises 2

The exercises have been split into key and extra exercises: make sure you are comfortable with key exercises first as they cover important calculations or key geometric concepts.

We expect you to spend approx. 2 hours on exercises, don't worry about finishing them all.

1 Key Exercises

A lot of these questions are key calculations; as a result we've made a lot of them available but do not feel you have to attempt them all, especially if you are comfortable with the calculation.

Question 1 Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) orthonormal basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

(1) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_3$

(2) $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{f}_3 = \mathbf{e}_3$

(3) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{f}_3 = \mathbf{e}_3$

(4) $\mathbf{f}_1 = \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1$, $\mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ orthogonal?
[*Hint: You can use the transition matrices from Exercises 1, Question 3*]

Question 2 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for $\mathbb{R}^2$. Write down the matrix for the following linear operators in the basis $\mathcal{B}$.

(1)
$$\begin{aligned} P_1 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x \\ \mathbf{e}_y &\mapsto 3\mathbf{e}_y \end{aligned}$$

(2)
$$\begin{aligned} P_2 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x - \mathbf{e}_y \end{aligned}$$

(3)
$$\begin{aligned} P_3 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto -4\mathbf{e}_x - 2\mathbf{e}_y \end{aligned}$$

(4)
$$\begin{aligned} P_4 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto 2\mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_x + 2\mathbf{e}_y \end{aligned}$$

Question 3 Let $\mathcal{B}, \mathcal{C}$ be (ordered) bases of a vector space V. Show that the transition matrix ${}_{\mathcal{C}}T_{\mathcal{B}}$ from $\mathcal{C}$ to $\mathcal{B}$ is the inverse matrix of the transition matrix ${}_{\mathcal{B}}T_{\mathcal{C}}$ from $\mathcal{B}$ to $\mathcal{C}$ i.e.

$${}_{\mathcal{C}}T_{\mathcal{B}} = ({}_{\mathcal{B}}T_{\mathcal{C}})^{-1}.$$

Question 4 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an orthonormal basis for $\mathbb{E}^3$. Calculate the determinant and trace for each of the following linear operators:

(1)
$$\begin{aligned} P_1 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x + \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x + \mathbf{e}_z \end{aligned}$$

(2)
$$\begin{aligned} P_2 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x - \mathbf{e}_y \\ \mathbf{e}_y &\mapsto \mathbf{e}_y - \mathbf{e}_z \\ \mathbf{e}_z &\mapsto \mathbf{e}_x - \mathbf{e}_z \end{aligned}$$

(3)
$$\begin{aligned} P_3 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \mathbf{e}_x \\ \mathbf{e}_y &\mapsto -\mathbf{e}_z \\ \mathbf{e}_z &\mapsto -\mathbf{e}_y \end{aligned}$$

(4)
$$\begin{aligned} P_4 \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{e}_x &\mapsto \sqrt{2}\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z \\ \mathbf{e}_y &\mapsto -\sqrt{2}(\mathbf{e}_y + \mathbf{e}_z) \\ \mathbf{e}_z &\mapsto \sqrt{2}\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z \end{aligned}$$

2 Extra Exercises

Question 5 Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be vectors in $\mathbb{E}^3$ such that $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, and $\mathbf{c}$ has length $\sqrt{3}$ and forms the angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with $\mathbf{a}$ and $\mathbf{b}$.

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ forms an orthonormal basis for $\mathbb{E}^3$.

Question 6 Let $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be vectors of $\mathbb{E}^3$ such that each has unit length and they are pairwise orthogonal with each other.

(1) Show that they are linearly independent.

(2) Show that $\mathcal{C}$ forms a basis for $\mathbb{E}^3$.

Question 7 Let $P\colon V \to V$ be a linear operator acting on the vector space V and let $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis for V. Prove that for any $\mathbf{v} \in V$, the column vector representing the image of $\mathbf{v}$ in the $\mathcal{B}$ basis, $[P(\mathbf{v})]_{\mathcal{B}}$, is given by the matrix multiplication: $[P]_{\mathcal{B}} [\mathbf{v}]_{\mathcal{B}}$.

You may use without proof the fact that matrix multiplication is linear: that is, for a matrix M, column vectors v, w and scalars λ, μ the following holds:

$$M(\lambda v + \mu w) = \lambda M v + \mu M w$$

Question 8 Let P be a linear operator acting on 2-dimensional vector space V and let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be a basis for V, such that the matrix of P in the basis $\mathcal{B}$ is given by

$$[P]_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}.$$

Show that there is a basis $\mathcal{C} = (\mathbf{e}_u, \mathbf{e}_v)$ such that the linear operator in the basis $\mathcal{C}$ is

$$[P]_{\mathcal{C}} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$

Hint: *The second matrix means that $P(\mathbf{e}_u) = 4\mathbf{e}_u$ (it is an eigenvector). Use this fact to equate $[P(\mathbf{e}_u)]_{\mathcal{B}}$ with $[4\mathbf{e}_u]_{\mathcal{B}}$.*

Question 9 Let $\mathcal{B} = (\mathbf{e}_x, \mathbf{e}_y)$ be an orthonormal basis for 2-dimensional Euclidean space $\mathbb{E}^2$.

(1) Consider the following alternative bases:

(a) $\mathcal{C}_{1a} = (\mathbf{e}_x, -\mathbf{e}_y)$ (b) $\mathcal{C}_{1b} = (\frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}, \frac{\mathbf{e}_y - \mathbf{e}_x}{\sqrt{2}})$

In each case write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}_\bullet$ and calculate the transition matrix from $\mathcal{C}_\bullet$ to $\mathcal{B}$.

(2) For each of the linear operators $P_{(1)}$ and $P_{(2)}$ from question Question 2 calculate the matrix for the linear operator in each $\mathcal{C}$ basis above.

Question 10 For each linear operator in Question 4:

(1) Determine if the operator is orthogonal or not.

Note: *It may be useful to recall a fact we learnt about the determinant of an orthogonal operator.*

(2) For $\lambda \in \mathbb{R}$ define the scaled linear operator $S\colon \mathbb{E}^3 \to \mathbb{E}^3$, by $S(\mathbf{v}) = \lambda P(\mathbf{v})$. Determine the values of λ (if any exist) such that S is an orthogonal operator.